THE IMPACT OF OPTIONS ON INVESTMENT PORTFOLIOS IN THE SHORT-RUN AND THE LONG-RUN, WITH A FOCUS ON DOWNSIDE PROTECTION AND CALL OVERWRITING

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Abstract. In this article I analyse the impact of the introduction of options on an investment portfolio. In particular I derive closed-form formulae for the standard measures of portfolio efficiency: risk-premium, risk, Sharpe ratio, and beta, of any portfolio containing any combination of options. Using these formulae on three examples of simple option strategies (call-overwriting, put protection, and collars) I show how these statistics are altered by the inclusion of an option overlay on a portfolio. Furthermore, I show that if an option strategy is repeated over multiple investment time-periods, the long-run return becomes normally distributed. The objective of this article is to provide investors with the mathematics to assess the impact of the introduction of options on portfolio efficiency, and to encourage a potential portfolio rebalance to account for this impact. Then to highlight that whilst options can create asymmetric, non-normal, outcomes, their repeated use may not alter the long-run portfolio return in the desired way and thus to encourage investors to assess if an option overlay will deliver the desired long-run outcome.

1. Introduction

Options are derivative instruments which alter the characteristics of an underlying asset. They can be added to an investment portfolio to engineer desired outcomes, in some sense improving the portfolio efficiency, for example seeing widespread use in providing downside protection and enhancing the income of a portfolio.

Options specialists measure risk using the so-called Greeks (e.g., *delta*, *gamma*, *theta*, *vega*)[1](pp. 298–307), which are not the conventional measures of portfolio efficiency in the asset management industry. The latter typically measure a portfolio's future return distribution with the three summary statistics *expected return* and *risk*, and the portfolio *beta* (the return sensitivity of the portfolio relative to the market)[2]. Consequently, options are used in asset management with scant regard for their impact on these summary statistics.

In the first half of this article, I introduce options then define the return of a portfolio involving options, being careful to avoid ambiguity on cost and leverage. I then provide closed form expressions for these summary statistics (expected return, risk, and beta) for a portfolio containing *any* option strategy. I apply these formulae in three worked examples of popular option strategies: income enhancement (*call-overwriting*), downside protection (*put-protection*), and reduced cost downside

Date: April 2022.

2010 *Mathematics Subject Classification.* Primary 60E05.

Key words and phrases. risk-premium; risk; Sharpe ratio; beta; call option; put option; put-protection; call-overwriting; collars.

protection (a *collar strategy*). I show that the beta of the portfolio is reduced by the inclusion of options. I further show in these examples that both the expected return and risk are also reduced, but at differing rates so that the Sharpe Ratio of the portfolio is impacted by these option strategies. That is to say, the efficiency of portfolios may be reduced by the introduction of options. The materiality of these reductions depends significantly on the option strike prices.

My motivation is to encourage portfolio managers to i) be judicious in the choice of strike prices in light of the influence of strike price on return, risk, and beta; ii) assess if the investors' return, risk, and beta targets are still met when options are deployed, and iii) rebalance the portfolio accordingly if summary statistic targets are missed as a consequence of overlaying an option strategy.

The time horizon is important when options are used in investment portfolios: an option strategy is over a single period, which is typically three months long and seldom greater than a year, whereas the horizon of long-term investors is significantly longer. For example, pension funds have triennial valuations and can invest with a time horizon far longer even than that. In the second part of this article, I consider the long-term return distribution of an investment portfolio which includes an option strategy that is sequentially repeated (a quarterly call-overwriting strategy repeated twelve times in a three-year period, would be an example). I define when two option strategies are identical, then prove that if an option strategy is sequentially repeated, and the strategies are all identical, then the long-term investment portfolio return tends to a normal distribution. Using simulations, I demonstrate this result for my three example strategies, showing that the convergence to a normal distribution is fast, but does depend on the strike price.

My motivation for presenting this long-run result is two-fold. First, as measures of portfolio efficiency, the summary statistics of expected return, risk, and beta are used to construct and assess investment portfolios. That is only theoretically valid if the portfolio return is normally distributed. The use of these summary statistics is considered appropriate on portfolios of conventional assets (such as equities and bonds) because empirical evidence suggests that returns of these assets appear close to being normally distributed, especially for holding periods of a month or more[3](and references therein). However, given that options specifically create asymmetry, single investment period returns of portfolios containing options will be non-normal and these statistics may not be valid in this case[20]. Showing that returns of portfolios containing option strategies tend to a normal distribution over time implies the summary statistics can be used over longer investment horizons, even if asymmetry is present in the short term. Second, I use this result to caution portfolio managers against sequentially repeating option strategies if portfolio efficiency is taken to be the asymmetry of return over the longer term.

2. Options

2.1. **Definitions relating to options.** A *call option* on an underlying security is the right, but not the obligation, to buy that security at a pre-set price on a specified future date. The pre-set price is termed the *strike price*, and the specified future date is termed the *expiry date*. The value of the call option on the expiry date is $\max(x - k, 0)$ where k denotes the strike price, and x denotes the value of the underlying security on the expiry date of the option (I will use this denotation of x for the remainder of this article). Note that since the value of the call option at expiry is never negative, but sometime positive, the call option must have non-zero cost to purchase.

Whereas a call option is the right to buy an underlying security, a *put option* is the right to sell that security.

An option is visually represented by a graph of the value of the option at expiry as a function of the underlying asset price (termed a *payoff profile*[4]). The payoff profiles for a call and put option are shown in Figure 1.

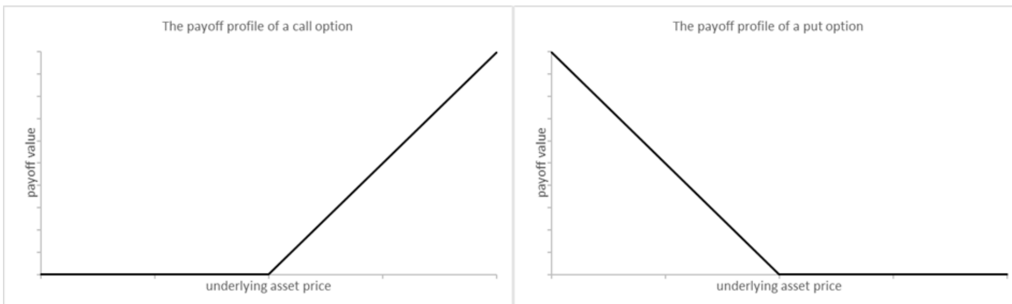

FIGURE 1. The payoff profiles of call and put options.

The difference of between the strike price and the underlying asset price, as a proportion of the underlying asset price is termed the option's *moneyness*. If the strike price is less than the asset price, a call option is said to be *in-the-money*; if the strike price is more than the asset price a call option is *out-of-the-money* (for put options the reverse is true). For example, a call option with a strike of 95 on an underlying asset with price of 100, is 5% in-the-money. Options with a strike price equal to the asset price are termed *at-the-money*.

2.2. **Examples of option strategies.** To demonstrate the concepts in this article, I will apply them to three examples of commonly used option strategies:

(1) *Call-overwriting* is the selling of a call option on an underlying asset that is already owned. The investor receives the price of a call option in lieu of a sacrifice of underlying asset price appreciation beyond a specified price. Call-overwriting is commonly used to boost income from the underlying security, which is usually either an individual equity security or an equity index.[5]
(2) *Put-protection* is the buying of a put option on an underlying asset that is already owned. The option price is paid by the investor in exchange for a guaranteed minimum underlying asset price on the expiry date, akin to an insurance payment. Put-protection is often applied to equity indices, where the guarantee is set to be the purchase price of the index thereby providing capital protection.[5]
(3) *Collar*: a combination of put-protection and call-overwriting whereby protection is sought but upside is sacrificed to offset the cost of protection.[6]

The payoff profiles of these three strategies are shown in Figure 2. The first panel shows the payoff profile of call-overwriting overlaid on a portfolio containing the underlying asset. We can see that this payoff profile is identical to selling a put option. The second panel shows the payoff profile of put-protection overlaid on a portfolio containing the underlying asset. We can see that this payoff profile is identical to buying a call option. The third panel shows the payoff profile of a collar, where both underlying asset price appreciation and deprecation are capped. This strategy can be overlaid on an underlying asset by buying a put or selling a call buying a call and selling a put, or constructed by buying a call and selling a put.

Generally, the underlying asset can be considered separately, with an option strategy overlaid, or the underlying asset can be considered as part of the option strategy. Mathematically, these are equivalent, and, in this article, I will express strategies in either way to facilitate readability.

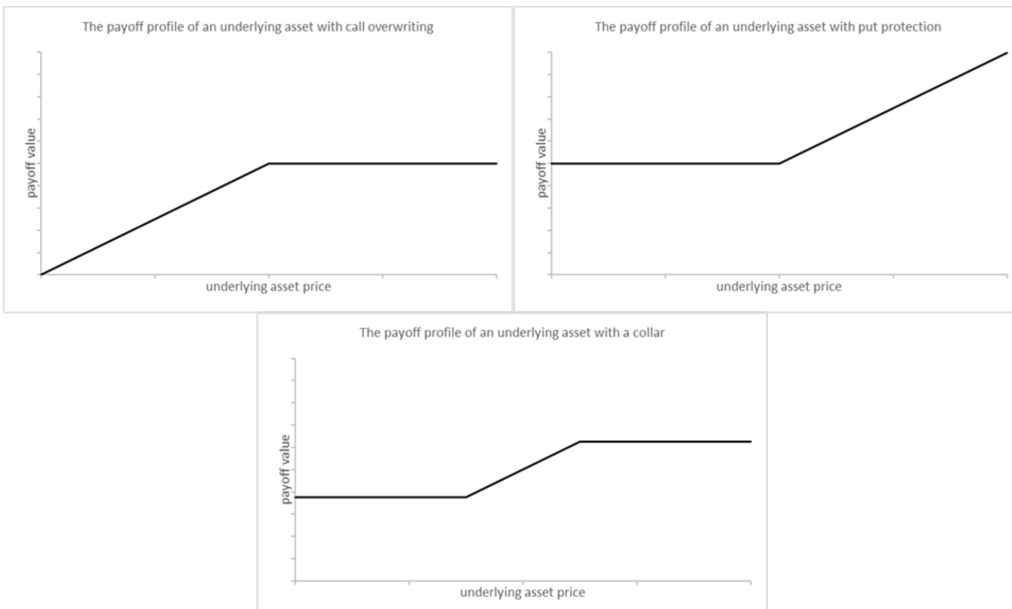

FIGURE 2. The payoff profiles of underlying asset overlaid with: (**a**) a call-overwriting strategy is the first panel, (**b**) a put-protection strategy in the second panel, and (**c**) a collar strategy in the third panel.

2.3. **Portfolios including options.** This article centres around the concept of using options to engineer the payoff profile of a portfolio. I will hereafter, except for Section 3.4, assume there is one risky asset - this could be a single security or an entire market portfolio. To simplify the mathematics of this article I have assumed non-dividend paying underlying assets. It is a routine algebraic exercise for the reader to incorporate dividends into the formulae shown here.

I also assume that a risk-free asst exists. Therefore, in our context a traditional portfolio comprises solely of the risk-free asset and the risky asset. To distinguish portfolios containing options, I define an *option adjusted portfolio* to refer specifically to traditional portfolios also contain options on the risky asset. The option adjusted portfolio can contain multiple options, with potentially differing strike prices, but for mathematical convenience I will restrict them to all having the same expiry date. I show in the Appendix that the payoff profile of *any* option adjusted portfolio is a continuous piecewise linear function of the underlying security price at the time of expiry. Letting $x_1, \ldots, x_n$ denote the strike prices of n options (ordered such that $x_1 < x_2 < \ldots < x_n$) and using x_0 and x_{n+1} to denote the upper and lower limits of the underlying asset price, hence $x_0 = 0$ and $x_{n+1} = \infty$, I can write the option adjusted portfolio value at expiry, denoted by $v(x)$, as

$$v(x) = a_i + b_i(x - x_i) \qquad \text{for } x_i \leq x < x_{i+1} \tag{2.1}$$

where $a_0, \ldots, a_n$ are the intercepts, and $b_0, \ldots, b_n$ are the gradients of each segment. Specifically, a_0 represents the value at expiry of the cash holding; b_0 represents the number of the risky asset units held in the portfolio; and $b_1, \ldots, b_n$ are governed by the quantities of the options where b_i equals b_{i-1} plus the number of units of the ith call option held in the portfolio. The intercepts $a_1, \ldots, a_n$ are calculated using the formula $a_i = a_{i-1} + b_{i-1}(x_i - x_{i-1})$. This piecewise linear representation has a useful application as it is often easier to sketch the desired payoff profile of a portfolio, then calculate what options need to be purchased to create that. For example I show in the Appendix that to replicate a payoff profile made up of linear segments joined at $x_1, \ldots, x_n$, with intercepts $a_0, \ldots, a_n$, and gradients $b_0, \ldots, b_n$, the investor can hold ($a_0 \div$ risk-free asset end value) units of the risk-free asset; hold b_0 units of the risky asset; and hold $(b_i - b_{i-1})$ units of a call option with strike price x_i (for $i = 1, \ldots, n$). I also show in the Appendix that our summary statistics are unaffected by a vertical shift of the payoff profile, and therefore we can assume without loss of generality that $a_0 = 0$.

The input parameters for our three example strategies are shown in Table 1.

TABLE 1. parameters defining three popular option adjusted portfolios.

Strategy	x_i	b_i	a_i
Call-overwrite	x_1 (the price beyond which all upside is sacrificed)	$b_0 = 1$	$a_0 = 0$
($n = 1$)		$b_1 = 0$	$a_1 = x_1$
Put-protection	x_1 (the price below which downside is protected)	$b_0 = 0$	$a_0 = 0$
($n = 1$)		$b_1 = 1$	$a_1 = 0$
Collar	x_1 (the price beyond which all upside is sacrificed)	$b_0 = 0$	$a_0 = 0$
($n = 2$)	x_2 (the price below which downside is protected)	$b_1 = 1$	$a_1 = 0$
		$b_2 = 0$	$a_2 = x_2 - x_1$

2.4. **Investment returns of option adjusted portfolios.** The *return* on an investment is the change in its price as a proportion of the initial investment value. Trivially, the return on the underlying asset, denoted by r_A, is

$$r_A = \frac{x}{s} - 1 \tag{2.2}$$

where s denotes the initial underlying security value.

The return on the option adjusted portfolio needs a more careful definition. In practice, the portfolio wealth is largely invested in the underlying asset and the option strategy is overlaid on that. However, it is mathematically irrelevant whether we consider the portfolio as containing an investment in the underlying asset and an option strategy overlayed on that, or an investment in the risk-free asset and an option strategy overlayed on that, where the investment in the underlying asset is subsumed in the option strategy (this is an extension of put-call parity where the payoff of a call and cash is the same of a put and the underlying asset[1](p. 163)). For algebraic brevity I assume the latter. This enables me to consider the portfolio as if it consisted of an investment in the risk-free asset, an option strategy, and a loan taken out to purchase the option strategy. In practice portfolios often contain some cash, which could be used to purchase options, or, if the portfolio is fully invested in the risky asset, the options could be purchased by borrowing capital. I will assume the same rate of interest exists for both lending and borrowing, to remove the complexity regarding the source of funds for option purchases.

Therefore, the return to the option adjusted portfolio with payoff profile $v(x)$, denoted by r_O, is expressed as

$$r_O = \frac{v(x)}{s} - \frac{p}{s}(1 + r_F) + r_F \tag{2.3}$$

where p denotes the cost of the option portfolio, r_F denotes the risk-free rate of return until the expiry date.

The choice of denominator in the return calculation of an option adjusted portfolio is somewhat arbitrary because the payoff profile can be scaled by any amount, by scaling each b_i. However, the unit of scale of each b_i is the quantity of units held in the underlying asset. Therefore, I divide by the unit price of the underlying asset when calculating portfolio return. Defining the option strategy return in this way ensures consistency: if the option strategy payoff profile is simply the value of the underlying asset, i.e. $v(x) = x$, then $p = s$, in which case $r_O = r_A$.

3. Summary statistics

At the start of the investment period, s is known but x is unknown and therefore r_A is a random variable. The purpose of this article is to relate the summary statistics of the option strategies to those of the underlying asset and I therefore assume that the expected return and variance of return of the underlying asset, which I denote by $\mathbb{E}[r_A]$ and $\mathbb{V}[r_A]$, are known.

Similarly, r_O is also a random variable. In the remainder of this section, I will derive expressions for the summary statistics of r_O. Specifically, I will be considering three commonly used statistics[2]:

(1) *annualised option adjusted portfolio risk-premium.* Defined as $(\mathbb{E}[r_O] - r_F)/t$
(2) *annualised option adjusted portfolio risk.* Defined as $\sqrt{\mathbb{V}[r_O]/t}$
(3) *beta of the option adjusted portfolio with respect to the underlying asset.* Defined as $\mathbb{COV}[r_O, r_A]/\mathbb{V}[r_A]$

where $\mathbb{E}[.]$, $\mathbb{V}[.]$, and $\mathbb{COV}[.]$ denotes the expectation, variance, and covariance operators respectively.

These statistics are annualised, and hence are divided by t, the time remaining until the option expiry date.

3.1. **Implied volatility.** The expected return of an option depends on the volatility of the underlying asset return. Therefore, the price of an option is also a function of this volatility. If we know the price of an option, we can deduce the level of volatility that must be implied by this price, termed the *implied volatility.* When computing the implied volatility of a series of call options, with differing strike prices, we typically observe that the implied volatility is higher for lower strike prices, termed the *volatility skew.*

At the time of writing, 3^{rd} December 2021, the implied volatilities on options on the S&P500 expiring on 31^{st} March 2022, using the closing prices quoted by CBOE, exhibited this characteristic as can be seen in Figure 3.

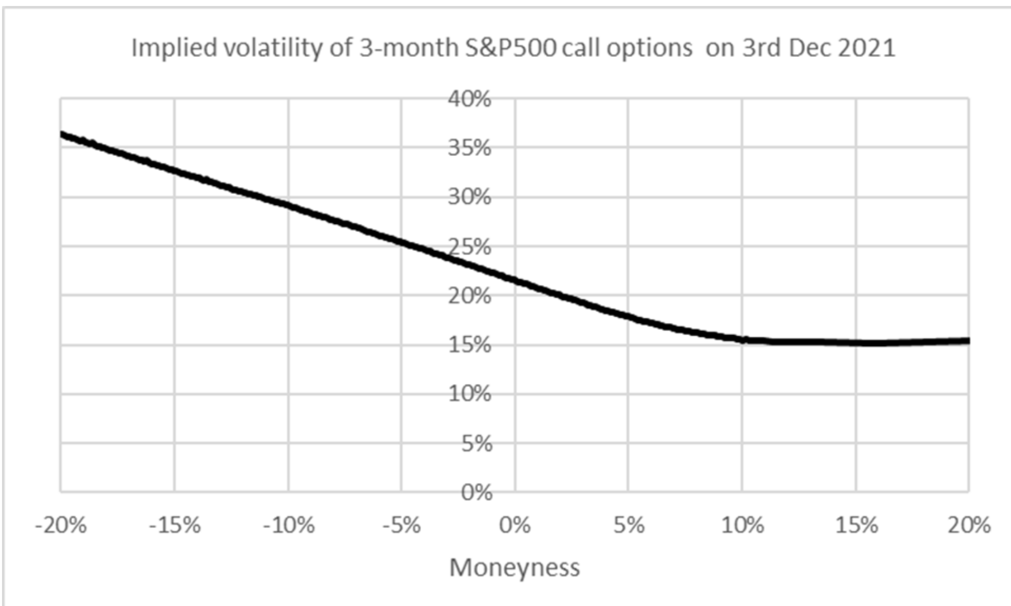

FIGURE 3. The implied volatilities on options on the S&P500 expiring on 31^{st} March 2022 using closing prices of 3^{rd} December 2021 quoted by CBOE. The typical volatility skew is clearly present where implied volatility increases as the strike price decreases.

This volatility skew accounts for uncertainty in the pricing models and reflects the risk averse nature of investors. Since both diminish in impact as the strike price of a call option increases, the implied volatility premium decreases as strike price increases, thus causing the skew.

A call option with very high strike price is extremely unlikely to be exercised, thus the model risk in pricing such an option is minimal. Therefore, the implied volatility of deep out-of-the-money options could be considered as the market's estimate of the volatility of the asset. In this case, there is always an increasingly large premium to be paid by the purchaser of a call option (or received by the seller of it) as the strike price decreases.

In the worked examples I will show how material this volatility skew is on the risk-premium of an option adjusted portfolio. After suggesting a typical value for actual volatility, I will estimate the corresponding implied volatilities by adding an *implied volatility premium* equal to the difference between the implied volatility as of 3^{rd} December 2021 and the minimum implied volatility across all strike prices on that day.

Implied volatility does not affect the risk and beta statistics, but does affect the risk-premium. It is conceptually obvious that the cost of a call option will equal the underlying asset price when its strike price is zero and decrease to zero as the strike price increases. However, the volatility skew causes the price to decrease more slowly when the strike price is less than the asset price, thereby rendering those options more expensive than if the volatility skew was not present. To illustrate the impact of the skew on risk-premium and Sharpe ratio (defined as the ratio of the risk-premium to risk) I will plot these statistics including and excluding the implied volatility premium.

3.2. **The summary statistics of option adjusted portfolios.** My first key result in this article is that the three statistics related to option adjusted portfolio return can be written in closed form as

$$\textit{risk-premium} = \frac{A + B - (A^* + B^*)}{t} \tag{3.1}$$

$$\textit{risk} = \sqrt{\frac{C + D + 2E - (A + B)^2}{t}} \tag{3.2}$$

$$\textit{beta} = \frac{F + G - Y(A + B)}{Z - Y^2} \tag{3.3}$$

where

$$Y = \exp\{\mu t\} \qquad \text{and} \qquad Z = \exp\{(2\mu + \sigma^2)t\},$$

and where

$$A = \sum_{i=0}^{n} A_i, \quad B = \sum_{i=0}^{n} B_i, \quad \ldots, \quad G = \sum_{i=0}^{n} G_i$$

with

$$\begin{aligned}
A_i &= y_i K_i \\
B_i &= b_i(J_i - q_i K_i) \\
C_i &= y_i^2 K_i \\
D_i &= b_i^2(L_i - 2q_i J_i + q_i^2 K_i) \\
E_i &= y_i b_i(J_i - q_i K_i) \\
F_i &= y_i J_i \\
G_i &= b_i(L_i - q_i J_i)
\end{aligned}$$

where $y_i = \frac{a_i}{s}$ and $q_i = \frac{x_i}{s}$. Also

$$
\begin{aligned}
K_i &= \Phi(u_{i+1}) - \Phi(u_i) \\
J_i &= Y(\Phi(v_{i+1}) - \Phi(v_i)) \\
L_i &= Z(\Phi(w_{i+1}) - \Phi(w_i))
\end{aligned}
$$

where

$$
\begin{aligned}
u_i &= \frac{\log(q_i) - (\mu - 1/2\, \sigma^2)t}{\sigma\sqrt{t}} \\
v_i &= \frac{\log(q_i) - (\mu + 1/2\, \sigma^2)t}{\sigma\sqrt{t}} \\
w_i &= \frac{\log(q_i) - (\mu + 3/2\, \sigma^2)t}{\sigma\sqrt{t}}
\end{aligned}
$$

with

$$
\mu = \frac{1}{t}\log(1 + \mathbb{E}[r_A]) \qquad \text{and} \qquad \sigma^2 = \frac{1}{t}\log\left(1 + \frac{\mathbb{V}[r_A]}{(1 + \mathbb{E}[r_A])^2}\right) \tag{3.4}
$$

$\Phi(.)$ denotes the standard normal cumulative distribution function, i.e.

$$
\Phi(z) = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} \exp\{-1/2\, x^2\} dx.
$$

A^* and B^* are to be computed similarly to A and B, but with μ replaced with r_f, defined as

$$
r_f = \frac{1}{t}\log(1 + r_F)
$$

and with σ replaced with the market's implied volatility of the options.

I show the derivations of these statistics in the Appendix.

3.3. **Comments on these general formulae.** I have assumed the Black Scholes method to price the options[1](p. 207), if readers want to use their own option price, then it can be substituted in place of the expression $A^* + B^*$ which represents the future value of option price at expiry, expressed as a proportion of the underlying asset price.

The option price only affects the risk-premium. It does not enter the calculation of risk or beta. Both of those measures are independent of option cost, and therefore are robust with respect to the pricing model used to price options.

When $i = 0$, u_i, v_i, and w_i are infinitely negative and when $i = n + 1$ they are infinitely positive. This causes no computational problems because these three variables only appear in the form $\Phi(u_i)$, $\Phi(v_i)$, and $\Phi(w_i)$ and the function $\Phi(.)$ is well behaved at $\pm\infty$ where $\Phi(-\infty) = 0$ and $\Phi(\infty) = 1$.

Both risk and beta are independent of the risk-free rate. Apart from the option price element of the risk-premium expression, the risk-free rate is not used in the risk-premium either. If the future values of the strike prices are initially specified, then discounted at the risk-free rate to identify $x_1, \ldots, x_n$, then even the risk-premium is independent of the risk-free rate.

It is not strictly necessary to assume Geometric Brownian motion. Only variables J, K, L, Y, and Z, depend on the distribution of asset prices and any distribution can be adopted to compute them (although the cost of the option embedded in the risk-premium expression should be checked for its validity if something other than Brownian motion is used).

3.4. **A universe containing multiple risky assets.** In practice, the investment universe will usually consist of multiple risky assets. Option strategies can be applied on an asset-by-asset basis rather than on the entire portfolio. This is especially the case for call-overwriting. Analogous to my portfolio nomenclature above, I term an individual asset which has an option overlay an *option adjusted asset*.

3.4.1. *Risk-premium.* In the case of option strategies being applied to single assets, the portfolio risk premium is given by

$$\textit{Portfolio risk-premium} = \sum_{i=1}^{N} w^{(i)} \textit{risk-premium}^{(i)} \tag{3.5}$$

where $w_{(i)}$ denotes the weight allocated to the ith asset (possibly option adjusted) and $\textit{risk-premium}^{(i)}$ denotes the risk premium of the ith asset. N denotes the number of assets in the investment universe. The individual risk-premium statistics can be computed as specified in the general expression in Section 3.2.

3.4.2. *Risk.* Portfolio risk, is defined as

$$\textit{Portfolio risk} = \sqrt{\sum_{i=1}^{N} w^{(i)2} \textit{risk}^{(i)2} + 2\sum_{i=1}^{N-1}\sum_{j=i+1}^{N} w^{(i)} w^{(j)} \mathrm{cov}^{(ij)}} \tag{3.6}$$

where $\textit{risk}^{(i)}$ denotes the risk of the ith asset, and $\textit{cov}^{(ij)}$ denotes the covariance of the returns of asset i and asset j. The individual risk statistics, $\textit{risk}^{(1)}, \ldots, \textit{risk}^{(N)}$, can be computed as specified in the general expression in Section 3.2. However, return covariances between each pair of assets are required to compute portfolio risk. The calculation of these covariances depends on whether one, or both, assets are option adjusted. I cover both cases below.

For notational ease, I will label the two assets in a pair, as asset 1 and asset 2. Every variable related to asset 1 will be identified by a superscript (1), and similarly by (2) for asset 2.

Following on from the single asset case, I will assume that the two asset prices follow Geometric Brownian motion with instantaneous means, $\mu^{(1)}$ and $\mu^{(2)}$, and instantaneous volatilities $\sigma^{(1)}$ and $\sigma^{(2)}$, but also with instantaneous correlation ρ. The value for ρ can derived from the covariance of the returns of the two underlying assets, denoted by $\mathbb{COV}[r_A^{(1)}, r_A^{(2)}]$, which I assume is known, using the expression

$$\rho = \frac{\log(1 + \mathbb{COV}[r_A^{(1)}, r_A^{(2)}])}{\sigma^{(1)}\sigma^{(2)} t} \tag{3.7}$$

Case i)

If asset 1 is option adjusted but asset 2 is not, the annualised covariance of their returns is

$$cov(ij) = \frac{D + E - (A + B)Y^{(2)}}{t} \tag{3.8}$$

where A and B are specified in the general expression in Section 3.2. However, D and E are now defined as follows:

$$D = \sum_{i=0}^{n} D_i \qquad \text{and} \qquad E = \sum_{i=0}^{n} E_i \tag{3.9}$$

with $D_i = b_i^{(1)}(L_i - q_i^{(1)} J_i)$ and $E_i = y_i^{(1)} J_i$ for $i = 0, \ldots, n$.

$$J_i = Y^{(2)}(\Phi(\nu_{i+1}) - \Phi(\nu_i)) \qquad \text{and} \qquad L_i = Y^{(12)}(\Phi(\xi_{i+1}) - \Phi(\xi_i))$$

where

$$\begin{aligned} \nu_i &= \frac{\log(q_i^{(1)}) - (\mu - {}^1\!/\!{}_2(\sigma^{(1)2} - 2\rho\sigma^{(1)}\sigma^{(2)}))t}{\sigma^{(1)}\sqrt{t}} \\ \xi_i &= \frac{\log(q_i^{(1)}) - (\mu + {}^1\!/\!{}_2(\sigma^{(1)2} + 2\rho\sigma^{(1)}\sigma^{(2)}))t}{\sigma^{(1)}\sqrt{t}} \end{aligned}$$

and where $Y^{(1)} = \exp\{\mu^{(1)}t\}$, $Y^{(2)} = \exp\{\mu^{(2)}t\}$, and $Y^{(12)} = Y^{(1)}Y^{(2)}\exp\{\rho\sigma^{(1)}\sigma^{(2)}t\}$.

Case ii)

If both assets have been option adjusted, the covariance can still be computed but the calculation is more complex. In the Appendix I show that the covariance of returns of two option adjusted assets is

$$cov(ij) = \frac{C + D + E1 + E2 - (A^{(1)} + B^{(1)})(A^{(2)} + B^{(2)})}{t} \tag{3.10}$$

The variables $A^{(1)}$, $A^{(2)}$, $B^{(1)}$ and $B^{(2)}$ are as specified in the general expression in Section 3.2 for asset 1 and asset 2. However, C and D, are now defined as

$$C = \sum_{i=0}^{n^{(1)}} \sum_{j=0}^{n^{(2)}} C_{ij} \qquad \text{and} \qquad D = \sum_{i=0}^{n^{(1)}} \sum_{j=0}^{n^{(2)}} D_{ij} \tag{3.11}$$

with $C_{ij} = y_i^{(1)} y_j^{(2)} K_{ij}$ and $D_{ij} = b_i^{(1)} b_j^{(2)} (L_{ii} - q_j^{(2)} J_{ij}^{(1)} - q_i^{(1)} J_{ij}^{(2)} + q_i^{(1)} q_j^{(2)} K_{ij})$. Also

$$E1 = \sum_{i=0}^{n^{(1)}} \sum_{j=0}^{n^{(2)}} E1_{ij} \qquad \text{and} \qquad E2 = \sum_{i=0}^{n^{(1)}} \sum_{j=0}^{n^{(2)}} E2_{ij} \tag{3.12}$$

with $E1_{ij} = y_i^{(1)} b_j^{(2)} (J_{ij}^{(2)} - q_j^{(2)} K_{ij})$ and $E2_{ij} = y_j^{(2)} b_i^{(1)} (J_{ij}^{(1)} - q_i^{(1)} K_{ij})$.

The assumption of geometric Brownian motion gives us that

$$
\begin{aligned}
K_{ij} &= \Phi_\rho(u^{(1)}_{i+1}, u^{(2)}_{j+1}) - \Phi_\rho(u^{(1)}_{i+1}, u^{(2)}_{j}) - \Phi_\rho(u^{(1)}_{i}, u^{(2)}_{j+1}) + \Phi_\rho(u^{(1)}_{i}, u^{(2)}_{j}) \\
J^{(1)}_{ij} &= Y^{(1)}(\Phi_\rho(v^{(1)}_{i+1}, \nu^{(2)}_{j+1}) - \Phi_\rho(v^{(1)}_{i+1}, \nu^{(2)}_{j}) - \Phi_\rho(v^{(1)}_{i}, \nu^{(2)}_{j+1}) + \Phi_\rho(v^{(1)}_{i}, \nu^{(2)}_{j})) \\
J^{(2)}_{ij} &= Y^{(2)}(\Phi_\rho(\nu^{(1)}_{i+1}, v^{(2)}_{j+1}) - \Phi_\rho(\nu^{(1)}_{i+1}, v^{(2)}_{j}) - \Phi_\rho(\nu^{(1)}_{i}, v^{(2)}_{j+1}) + \Phi_\rho(\nu^{(1)}_{i}, v^{(2)}_{j})) \\
L_{ij} &= Y^{(12)}(\Phi_\rho(\xi^{(1)}_{i+1}, \xi^{(2)}_{j+1}) - \Phi_\rho(\xi^{(1)}_{i+1}, \xi^{(2)}_{j}) - \Phi_\rho(\xi^{(1)}_{i}, \xi^{(2)}_{j+1}) + \Phi_\rho(\xi^{(1)}_{i}, \xi^{(2)}_{j}))
\end{aligned}
$$

where $\Phi_\rho(x, y)$ is the cumulative distribution function of the bivariate standard normal distribution with correlation ρ.

There are numerous approached to computing $\Phi_\rho(x, y)$. For example, it can be expressed as

$$
\Phi_\rho(x, y) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} \exp\{-{}^1\!/_2\, z^2\} \Phi\left(\frac{y - \rho z}{\sqrt{1-\rho^2}}\right) dz \tag{3.13}
$$

which can be solved by a one-dimensional quadrature routine.

3.4.3. *Beta.* In the single risky asset paradigm, I derived the beta of the option adjusted portfolio relative to the risky asset. When there are multiple risky assets, beta is more usually measured relative to some other reference asset – commonly a market index. Following my previous notational convention, I denote the first and second moment of this reference asset by $Y^{(m)}$ and $Z^{(m)}$ respectively, thus

$$
Y^{(m)} = \exp\{\mu^{(m)} t\} \qquad \text{and} \qquad Z^{(m)} = \exp\{(2\mu^{(m)} + \sigma^{(m)2})t\}, \tag{3.14}
$$

In the Appendix I show that the beta of an option adjusted asset relative to a reference asset is

$$
beta = \frac{D + E - (A + B)Y^{(m)}}{Z^{(m)} - Y^{(m)2}} \tag{3.15}
$$

where $D = \sum_{i=0}^{n} D_i$ and $E = \sum_{i=0}^{n} E_i$
with $D_i = b_i(L_i - q_i J_i)$ and $E_i = y_i J_i$ for $i = 0, \ldots, n$.

If the asset price and reference asset price have instantaneous correlation, $\rho^{(m)}$, then

$$
J_i = Y^{(m)}(\Phi(\nu_{i+1}) - \Phi(\nu_i)) \qquad \text{and} \qquad L_i = Y^{(1m)}(\Phi(\xi_{i+1}) - \Phi(\xi_i)) \tag{3.16}
$$

where

$$
\begin{aligned}
\nu_i &= \frac{\log(q_i) - (\mu - {}^1\!/_2(\sigma^2 - 2\rho\sigma\sigma^{(m)}))t}{\sigma\sqrt{t}} \\
\xi_i &= \frac{\log(q_i) - (\mu + {}^1\!/_2(\sigma^2 + 2\rho\sigma\sigma^{(m)}))t}{\sigma\sqrt{t}}
\end{aligned}
$$

and where $Y^{(1m)} = YY^{(m)} \exp\{\rho^{(m)}\sigma\sigma^{(m)} t\}$.

Note that $\rho^{(m)}$ can be derived from the covariance of the asset and the reference asset thus

$$\rho^{(m)} = \frac{\log(1 + \mathbb{COV}[r_A, r_M])}{\sigma\sigma^{(m)}t} \tag{3.17}$$

Once the individual asset betas have been computed, the portfolio beta is given as

$$\textit{Portfolio beta} = \sum_{i=1}^{N} w^{(i)} \textit{beta}^{(i)} \tag{3.18}$$

where $beta^{(i)}$ denotes the beta of the ith asset with respect to the reference asset.

4. Examples: single investment period

Let us now look at these formulae applied to our three examples. I will assume the underlying asset is an equity index and have therefore taken the annualised asset return, $\mathbb{E}[r_A]/t = 8\%$, and annualised asset risk $\sqrt{\mathbb{V}[r_A]/t} = 15\%$. I will consider quarterly strategies and hence will set $t = 0.25$. As stated earlier, the risk-free rate does not have a significant influence on the summary statistics. I set it as $r_F/t = 3.5\%$ which results in an annualised asset risk-premium of 4.5%. (My risk-premium and risk estimates are close to those published by the large asset managers, such as [7],[8], and [9]).

To incorporate the implied volatility, I have subtracted the implied vol curve as of 3rd Dec 2021 (from CBOE) from its minimum to derive an implied volatility premium, then added that to σ to generate an implied volatility curve for use I the examples below.

4.1. **Call-overwriting.**

A call-overwritten portfolio has only one strike, with a_1 equal to the strike price and b_0 equal to 1. All other variables have zero value. Therefore, the summary statistic expressions simplify significantly. They are

$$\textit{risk-premium} = \frac{A + B - (A^* + B^*)}{t} \tag{4.1}$$

$$\textit{risk} = \sqrt{\frac{C + D - (A + B)^2}{t}} \tag{4.2}$$

$$\textit{beta} = \frac{F + D - Y(A + B)}{Z - Y^2} \tag{4.3}$$

where $A = q\Phi(-u)$, $B = Y\Phi(v)$, $C = q^2\Phi(-u)$, $D = Z\Phi(w)$, and $F = qY\Phi(-v)$.

Note that for brevity I have dropped the subscripts from variables u, v, w, and q; they all refer to $i = 1$ in the general formulae. As stated in the general risk-premium formula, A^* and B^* are computed similarly to A and B but with μ replaced with r_f, and σ replaced with the implied volatility of the option.

Call-overwriting can be considered as the purchase of the underlying asset and the sale of a call option. Therefore, the implied volatility premium is in the investor's favour. Excluding that premium, the risk-premium from call-overwriting strategies is zero when the strike price is very low because the payoff is highly likely to be a fixed constant value. As the strike price increases, the upside sacrifice occurs at a higher price, and therefore the call-overwritten portfolio behaves more like the underlying asset. With a very high strike price, the call option is virtually indifferent from the underlying asset and therefore has the same risk-premium. This can be seen Figure 4.

However, the volatility skew results in a higher premium being harvestable for lower strike prices. The increase in risk-premium due to the implied volatility premium is dependent on the shape of the implied volatility curve, but using the data as of 3rd December 2021, the increase in risk-premium can be seen in Figure 4 to be significant for most strike prices. Indeed, for strike prices lower than the initial asset price, i.e. in-the-money options, where a guaranteed loss is made on the investment but the investor is given a premium in exchange for that loss, the risk-premium increase is huge. For example, the risk-premium of a call-overwritten strategy with an option struck 5% in-the-money would be less than 1% excluding the boost from the implied volatility premium but is more than 7% including it.

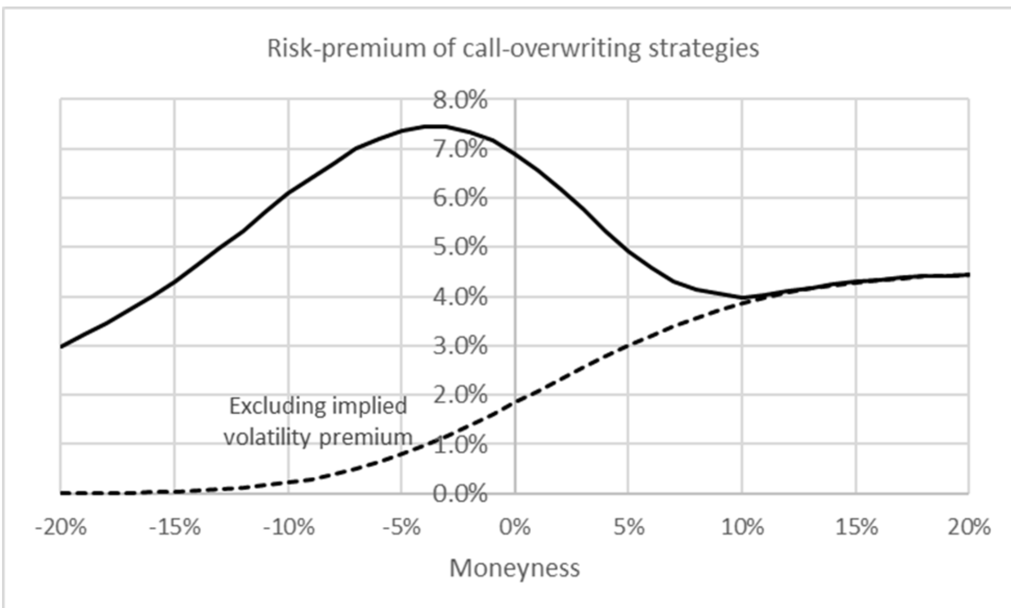

FIGURE 4. The risk-premium for an underlying asset with an overlaid call-overwriting strategy for a range of strike prices. The dotted line assumes the implied volatility is equal to the underlying asset volatility. The solid line uses includes the implied volatility premium in the price of the options.

Since the payoff is more likely to be a fixed constant as the strike price reduces, the risk of a call-overwriting strategy reduces. This can be seen in Figure 5 where the risk of the portfolio is less than half of the underlying asset risk for an at-the-money option. This risk reduction makes intuitive sense because in a call-overwriting strategy the price appreciation is capped, and thus there is no investment risk beyond that cap.

An important element of this paper is identifying the impact of option strategies on the portfolio Sharpe ratio. Whilst we have shown that, excluding the volatility premium, both the risk-premium and the portfolio risk both drop as strike price

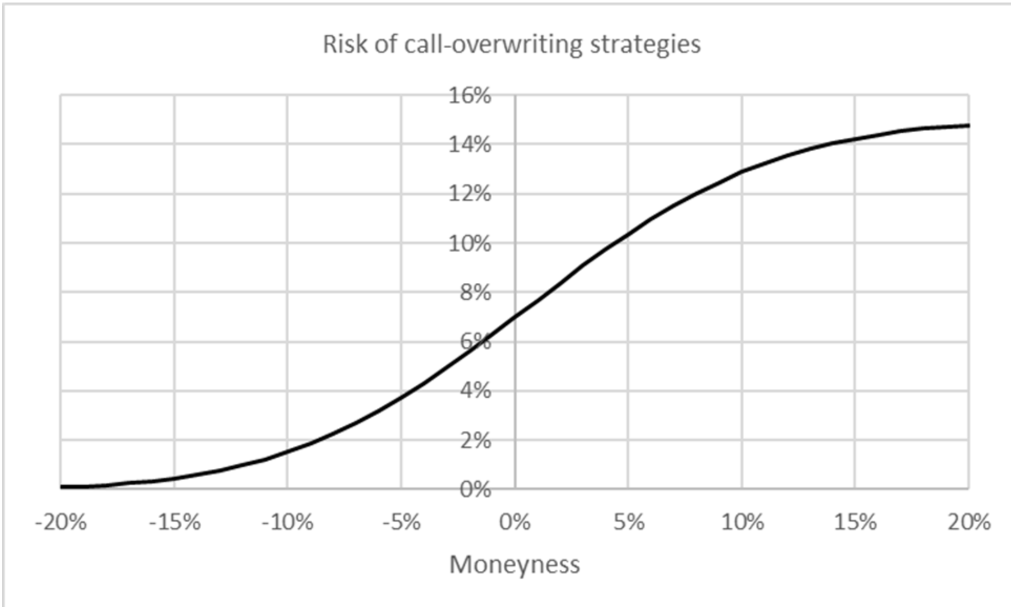

Figure 5. The risk of the return of an underlying asset with an overlaid call-overwriting strategy for a range of strike prices.

increases, they do not do so at the same rate. In fact, the risk-premium drops faster. Therefore, the Sharpe ratio, being the ratio of risk-premium to risk, also drops.

Figure 6 shows that this deterioration in Sharpe ratio is modest for out-of-the-money calls (.e with high strike prices) and is more than offset by the boost to risk-premium from the implied volatility premium. Figure 6 shows a significant increase of the Sharpe ratio when the implied volatility premium is included. For an at-the-money option the Sharpe ratio is approximately 1 and continues to rise sharply as the option is struck further in-the-money.

An option can be thought of as behaving like the underlying asset when its price is sufficiently high; and behaving like the risk-free asset when it is not. Consequently, the beta of the call-overwriting strategy drops as the strike price drops. This is shown in Figure 7 where the reduction in beta can be seen to be large for out-of-the-money options. For example, the beta of a quarter-year call-overwriting strategy when the option is struck 5% out-of-the-money has already dropped to nearly 0.6. For at-the-money options, the beta is only 0.3. Therefore, for investment strategies that target beta, a greater investment into the risky asset will be needed if a call-overwriting strategy is to be overlaid.

Call-overwriting is often used as an income enhancement tool by mutual funds[5], whereby the premium received for selling the call option is distributed, like a dividend. Some of the growth potential of the underlying asset is sacrificed in exchange for a payment and therefore, in effect, call-overwriting converts some of the capital appreciation into income. If the premium for call-overwriting is distributed as income, the expected capital appreciation of the remaining portfolio is

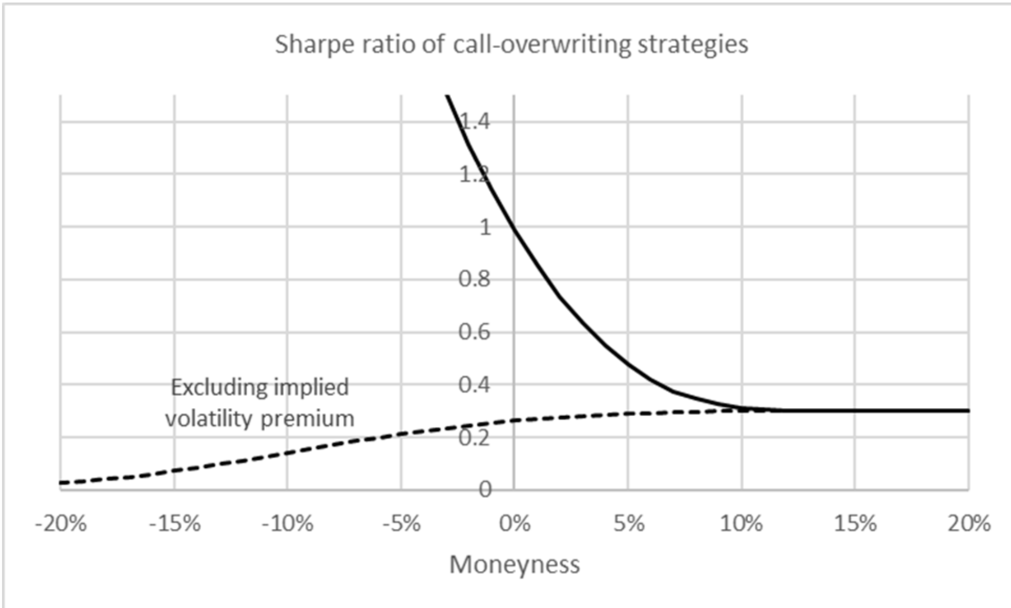

FIGURE 6. The Sharpe ratio for an underlying asset with an overlaid call-overwriting strategy for a range of strike prices. The dotted line assumes the implied volatility is equal to the underlying asset volatility. The solid line uses includes the implied volatility premium in the price of the options.

$$\textit{Expected capital appreciation} = A + B - 1 \tag{4.4}$$

Notice that the level of implied volatility does not affect the expected capital appreciation. The implied volatility only governs how much income is generated for a given strike price.

It is clear from this expression that the expected capital appreciation of a call-overwritten portfolio can be negative. Call-overwriting can turn the investment into a drawdown strategy. This may be a concern because such a portfolio will therefore inevitably become worthless over time. To see why this is so, consider a call-overwriting strategy where the call option is struck at-the-money. Then any asset price growth is sacrificed, but all asset price contraction is experienced. It is then inevitable that the expected capital appreciation is negative. The less of the asset growth is sacrificed, the higher the capital appreciation of the portfolio. This can be seen in Figure 8 where the expected annualised capital appreciation of a quarterly call-overwriting program is plotted.

The point at which the expected capital appreciation is zero can be identified by finding the strike price for which A + B = 1. This break-even strike price must clearly be greater than the initial asset price. In fact, when expected annualised asset return is 8% and its risk is 15%, this break-even strike price is 4.32% greater than the initial asset price, for quarter-year options. Any call-overwriting program with a strike price lower than this will cause the portfolio to become worthless if

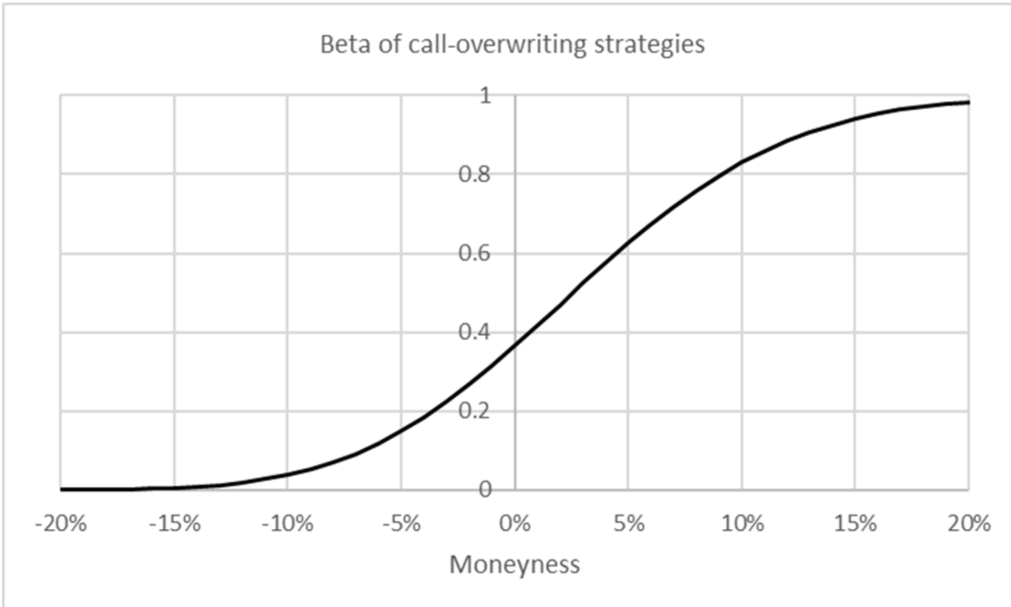

FIGURE 7. The price sensitivity (beta) of an underlying asset with an overlaid call-overwriting strategy for a range of strike prices, in relation to price change of the underlying asset.

the received price of selling the call option is distributed as income. This should be born in mind especially when harvesting large option prices by selling in-the-money options. Figure 9 shows how, for a simulated price time series of the underlying asset, the capital of three call-overwriting strategies behaves over time using quarterly options, struck 5% in-the-money, at-the-money, and 5% out-of-the-money. The 5% in-the-money overwriting has lost 75% of its initial value in 6 years.

4.2. **Put-protection.** A put-protected portfolio has only one strike. The payoff profile intercepts are both zero valued, and only b_1 is non-zero. Therefore, my general formulae simplify significantly. I can write that for a put-protected portfolio

$$\textit{risk-premium} = \frac{B - B^*}{t} \tag{4.5}$$

$$\textit{risk} = \sqrt{\frac{D - B^2}{t}} \tag{4.6}$$

$$\textit{beta} = \frac{G - YB}{Z - Y^2} \tag{4.7}$$

where Y and Z are defined as in the general formulae, and

$$\begin{aligned} B &= Y\Phi(-v) - q\Phi(-u) \\ D &= Z\Phi(-w) - 2Yq\Phi(-v) + q^2\Phi(-u) \\ G &= Z\Phi(-w) - Y\Phi(-v) \end{aligned}$$

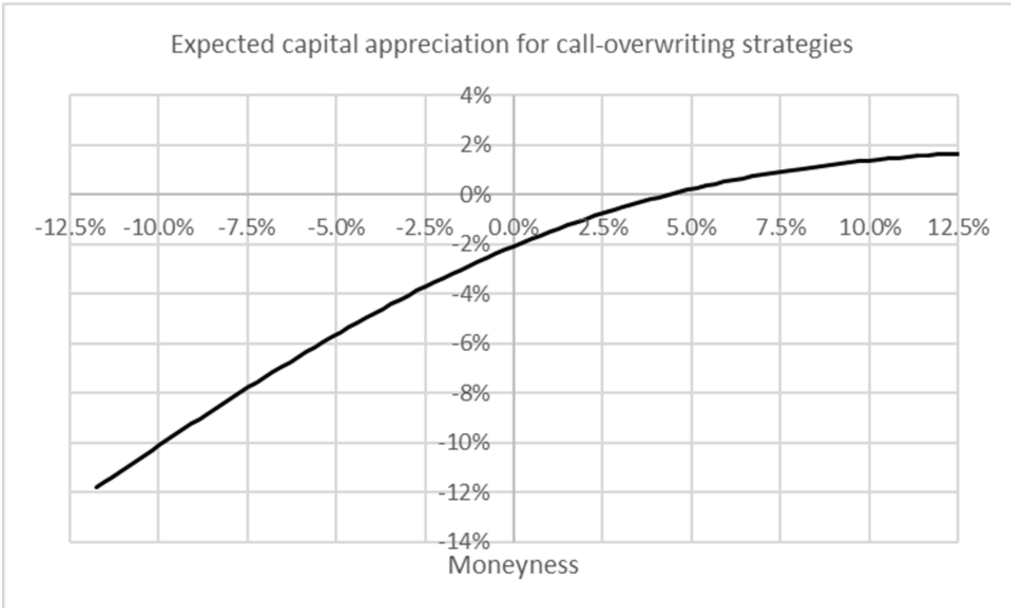

FIGURE 8. The expected amount of capital appreciation of a portfolio consisting of an underlying asset and a call-overwriting strategy for a range of strike prices, assuming the income from the sale of the call option is distributed to the investor. The expected appreciation is negative if the strike price is less than about 4% out-of-the-money. In this case the investment will eventually become zero valued.

Note that for brevity I have dropped the subscripts from variables u, v, w, and q; they all refer to $i = 1$ in the general formulae. As stated in the general risk-premium formula, B^* is computed similarly to B but with μ replaced with r_f, and σ replaced with the implied volatility of the option.

The higher the strike price in a put-protected portfolio, the more guaranteed the portfolio value is, and therefore the less risky it is. At an extreme, with infinitely high levels of protection, the portfolio is risk-free and therefore we'd expect the risk-premium of that portfolio to equal zero. Conversely, with no protection, the portfolio will behave identically to the underlying asset and should have the same expected return, excluding the impact of the implied volatility premium. I show in Figure 10, that we do indeed see this behaviour in our calculation of risk-premium. For example, at-the-money protection has a risk-premium less than 60% of the risk-premium of the underlying asset. However, the implied volatility premium has a significant effect on put-protection strategies. Using the CBOE implied volatility data on options with expiry at 31st March 2022 as of 3rd December 2021, Figure 10 illustrates that when the protection is desired against a loss of between 0% and 10%, the implied volatility is so high that the cost of protection causes the portfolio risk-premium to be negative.

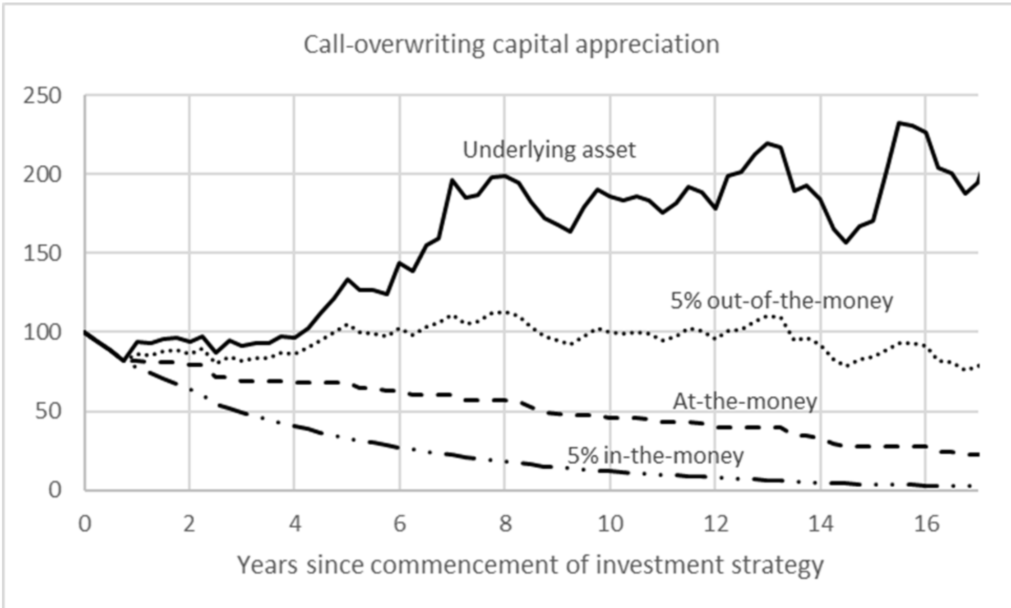

FIGURE 9. A simulation of the value of an underlying asset overlaid with a call-overwriting strategy for 3 strike prices: 5% out-of-the-money, at-the-money, and 5% in-the-money.

In summary, the introduction of put-protection lowers the expected return of a portfolio. The more protection, the more the expected return is lowered. However, the existence of an implied volatility skew can severely negatively impact this expected return.

It is intuitively obvious that the risk of the investment should drop as the guarantee increases. This effect is borne out mathematically and the risk reduction of our protection strategy as the strike price increases, as computed from our formula, is presented in Figure 11.

Excluding the implied volatility premium, as the moneyness decreases, the risk-premium of the put-protection strategy decreases faster than the risk, therefore the Sharpe ratio also drops. For an at-the-money strategy the drop is from 0.3 for the underlying asset to 0.25 for the protected strategy. However, the impact of the implied volatility skew is so severe on the risk-premium, that once included in the computation of Sharpe ratio, that ratio drops to -0.23. This is shown in Figure 12.

If the put-protection only occurs at a low strike price, then the put-protected portfolio will behave similarly to the underlying asset, and its beta will be close to 1. However, as the strike price rises, the protection begins to dominate and the beta decreases, eventually becoming zero. This behaviour can be seen in Figure 13. For protection less than 5% below market, the beta of the put-protected portfolio can be seen to be above 0.8. However, beta decreases rapidly thereafter, being about 0.6 for at-the-money protection, and less than 0.4 for 5% in-the-money protection.

4.3. **Collars.** A collar can be overlayed on an underlying asset, by buying one put and selling one call. Again, my general formulae simplify significantly for such a strategy. I can write that for a portfolio with a collar

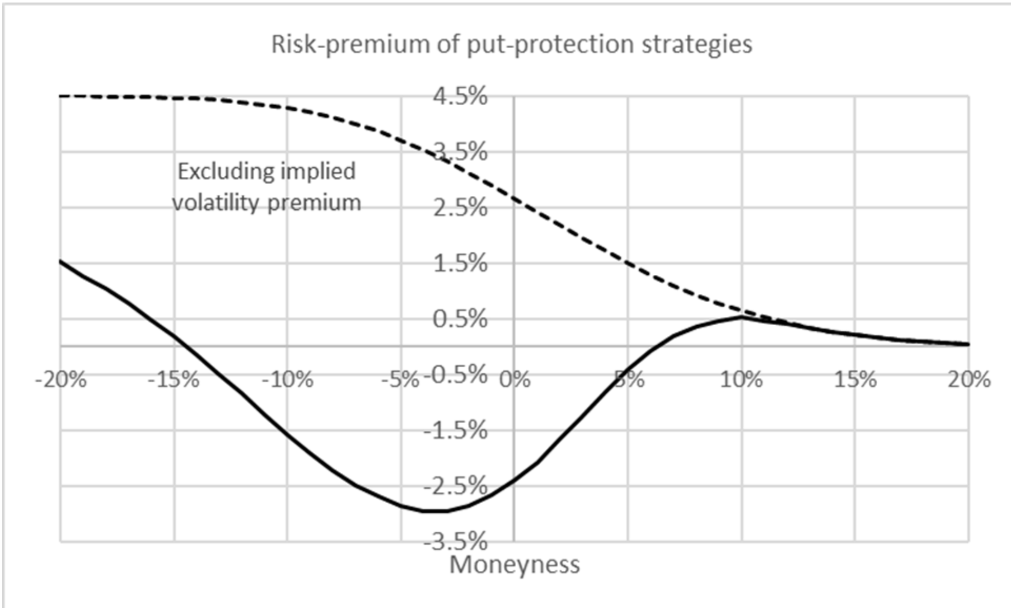

FIGURE 10. The risk-premium for an underlying asset with an overlaid put-protection strategy for a range of strike prices. The dotted line assumes the implied volatility is equal to the underlying asset volatility. The solid line uses includes the implied volatility premium in the price of the options.

$$\textit{risk-premium} = \frac{A + B - A^* - B^*}{t} \tag{4.8}$$

$$\textit{risk} = \sqrt{\frac{C + D - (A + B)^2}{t}} \tag{4.9}$$

$$\textit{beta} = \frac{F + G - Y(A + B)}{Z - Y^2} \tag{4.10}$$

where Y and Z are defined as in the general formulae, and

$$\begin{aligned}
A &= (q_c - q_p)\Phi(-u_c) \\
B &= Y[\Phi(v_c) - \Phi(v_p)] - q_p[\Phi(u_c) - \Phi(u_p)] \\
C &= (q_c - q_p)^2\Phi(-u_c) \\
D &= Z[\Phi(w_c) - \Phi(w_p)] - 2q_pY[\Phi(v_c) - \Phi(v_p)] + q_p^2[\Phi(u_c) - \Phi(u_p)] \\
F &= (q_c - q_p)Y\Phi(-v_c) \\
G &= Z[\Phi(w_c) - \Phi(w_p)] - q_pY[\Phi(v_c) - \Phi(v_p)]
\end{aligned}$$

The subscripts, p and c, of variables u, v, w, and q refer to the put and the call, respectively, and map to $i = 1$ and $i = 2$ respectively in the general formulae.

FIGURE 11. The risk of the return of an underlying asset with an overlaid put-protection strategy for a range of strike prices.

As stated in the general risk-premium formula, A^* and B^* are computed similarly to A and B but with μ replaced with r_f, and σ replaced with the implied volatility of the option.

A collar strategy is commonly used in practice as a substitute for put-protection[6]. Its cost is lower than the same level of put-protection because the purchase price of the put is offset by the sale of the call. Indeed, for any out-of-the-money put strike price, it is possible to select a call strike price such that the overall cost of the collar strategy is zero. That call strike price is the solution to

$$A^* + B^* = (1 + r_F)(1 - q_p) \tag{4.11}$$

The collar has conceptual merit for the investor in that a put-protection strategy pays the implied volatility premium whereas the collar also received this premium through the sale of the call. Therefore, the investor does not bear the full cost of this implied volatility premium in a collar strategy. However, in practice the volatility skew materially dilutes this benefit because a higher premium is paid for the put than is received for the call. This can be seen in Figure 14 where the call strike price is plotted against the put strike price in order to construct a zero-cost collar. The dotted line shows this relationship excluding the volatility skew where, very roughly, the put strike and call strike are seen to be equidistant from the current market price. For example, for put-protection 10% below current market price, only capital appreciation over 12% is sacrificed to fund the put. The solid line shows this relationship including the volatility skew, where the impact can be seen to be meaningful. In this case, for put-protection of 10% for example, all capital appreciation above 5% must be sacrificed.

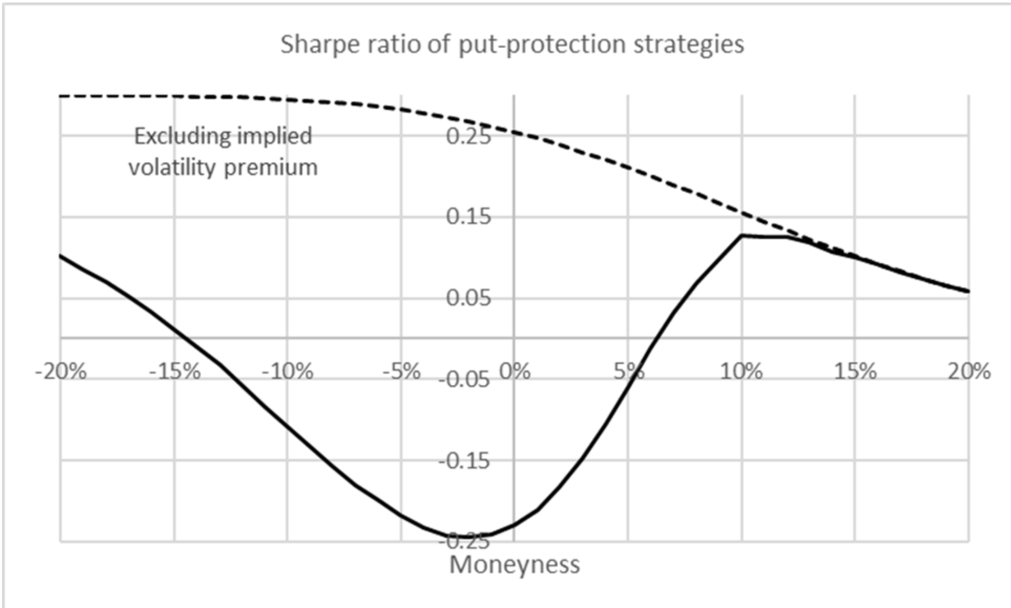

FIGURE 12. The Sharpe ratio for an underlying asset with an overlaid put-protection strategy for a range of strike prices. The dotted line assumes the implied volatility is equal to the underlying asset volatility. The solid line uses includes the implied volatility premium in the price of the options.

Given the impact of volatility skew on identifying the strike prices of zero-cost collars, I will consider symmetric collars. These are collars with the strike price of the call and the put being equidistant (in moneyness) from current underlying asset price (with the put strike below the current price, and the call strike above it).

As we saw above, both call-overwriting and put-protection reduce a portfolio's expected return, we should therefore expect the same for the collar. The fact that a collar costs less than a put (with the same strike price) should not be seen as implying a collar has less impact on the portfolio expected return. In fact, the cost of the collar has no direct bearing on expected return; it is the distance between the put and call strike prices that governs that. In Figure 15 I show the risk-premium of symmetric collar strategies. The dotted line shows that, excluding the effect of the implied volatility premium, if the put and call strike prices are far from the current price, then the risk-premium of the collar equals the risk-premium of the underlying asset. As the strike prices converge, the risk-premium decreases, becoming zero when the strike prices are identical. However, volatility skew reduces the risk-premium of a collar significantly. With the volatility skew prevalent at the time of writing, the Share ratio of the collar was negative for all but the most extreme choices of put and call strikes.

The monotonic drop in the symmetric collar risk-premium when excluding the effect of the volatility skew, is expected. When the strike prices are both far from the current underlying asset price, there is neither any protection, nor any capping

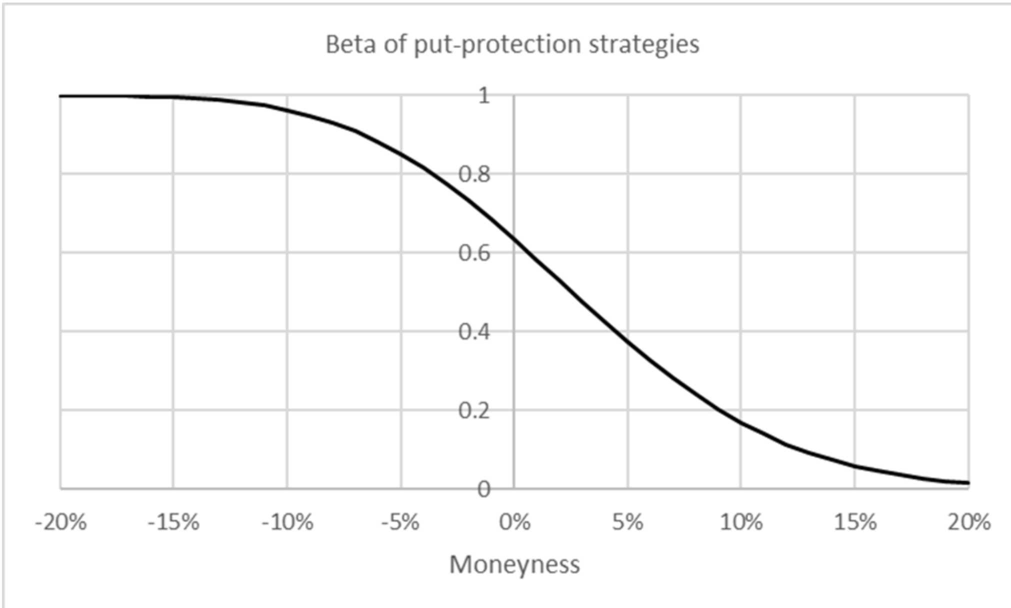

Figure 13. The price sensitivity (beta) of an underlying asset with an overlaid put-protection strategy for a range of strike prices, in relation to price change of the underlying asset.

of appreciation, therefore the risk of the collar will match the risk of the underlying asset. However, since the payoff from a collar strategy is a fixed constant when the underlying asset price is either less than the put strike or greater than the call strike, the risk of a collar strategy reduces as these two strike prices converge, eventually becoming risk-free when the strike prices are equal. I show this behaviour in Figure 16.

Ignoring the volatility skew, the reduction in risk-premium as the strike prices converge is similar to the reduction in risk. Therefore, Figure 17 shows that the Sharpe ratio remains nearly constant, at the same level as the underlying asset Sharpe ratio, until the strike prices within 5% of the current underlying asset price. This behaviour of the collar Sharpe ratio implies that the reduction in Sharpe ratio in the call-overwriting and put-protection strategies stems from the asymmetry in return distribution that the option strategies introduce. Nevertheless, the impact of the volatility skew on the risk-premium of a collar results in a poor Sharpe ratio, being negative Sharpe ratio for strike prices within 15% of the current underlying price.

Since both put-protection and call-overwriting reduce the beta of a portfolio, the combination on them in a collar also causes a reduction in beta. The combination actually causes this reduction to occur more rapidly for a collar than either put-protection or call-overwriting. This is shown in Figure 18. For example, while a 5% call-overwritten portfolio has a beta of 0.63 and a 5% put-protected portfolio has a beta of 0.85, Figure 18 shows that a 5% collar has beta of only 0.48. Therefore, investors should be mindful that if the allocation to the underlying asset was made

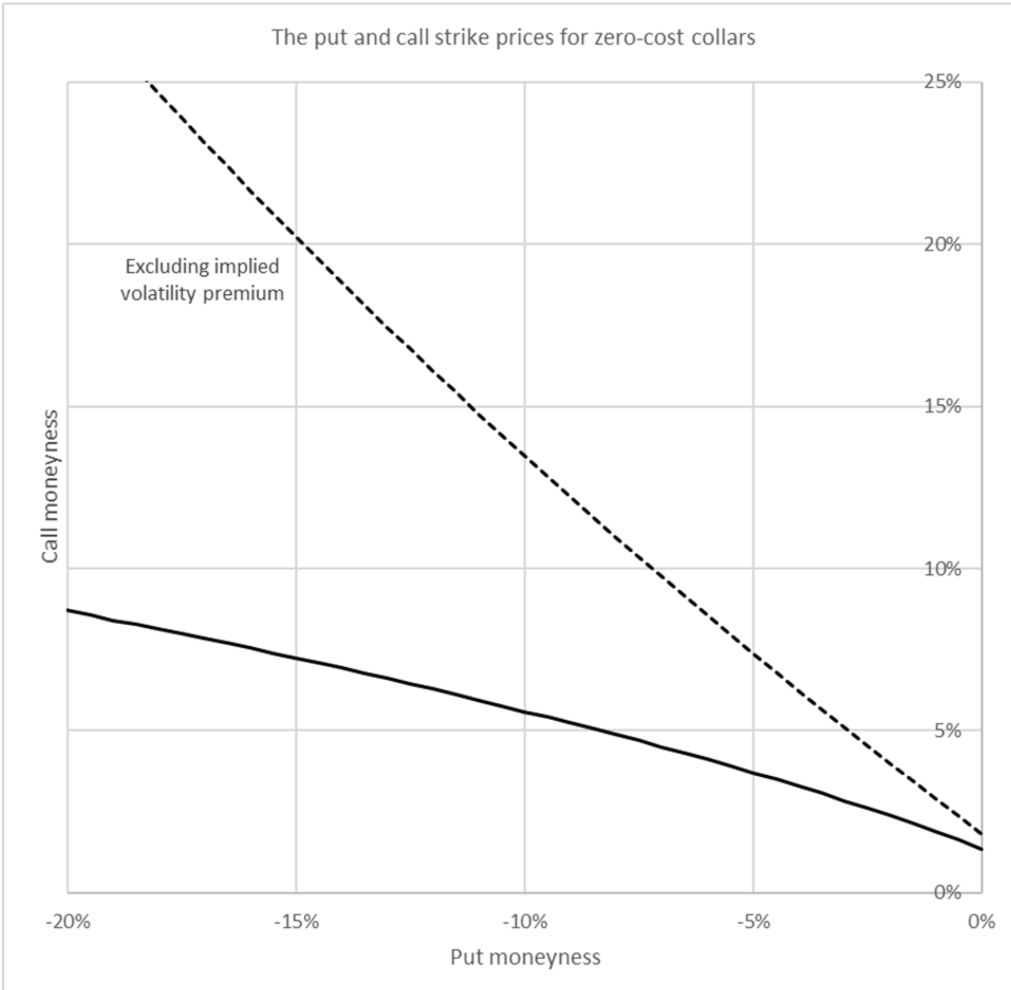

FIGURE 14. The strike price of call such that a collar has zero initial cost, for a range of put strike prices. The dotted line assumes the implied volatility is equal to the underlying asset volatility. The solid line uses includes the implied volatility premium in the price of the options. The put is more expensive when incorporating volatility skew, so that more capital appreciation must be sacrificed, resulting in a lower call strike price.

prior to the collar being applied, the beta of that portfolio will be less than half of the target amount if a 5% collar is then implemented.

5. Literature Review

There is a body of literature which endeavours to compute instantaneous risk statistics, notably [10]. This includes the measurement of instantaneous betas in [11]. However, my interest is not in the measurement of these statistics at a point in time in the life of the option, it is to provide measures over the life of the option.

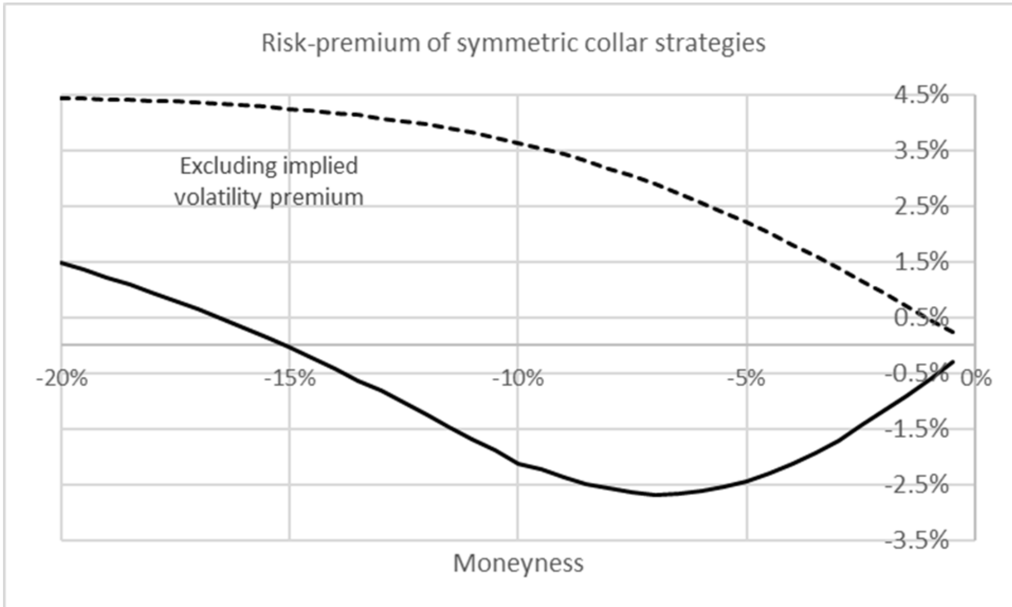

FIGURE 15. The risk-premium for an underlying asset with an overlaid symmetric collar strategy for a range of strike prices. The dotted line assumes the implied volatility is equal to the underlying asset volatility. The solid line uses includes the implied volatility premium in the price of the options.

Papers providing a theoretical framework, in the context I espouse, for deriving return and risk statistics are extant, but tend to focus on specific strategies. For example, Figelman[12] provides a formulation for covered call strategies.

There have been many empirical studies measuring ex-post risks and returns of option adjusted portfolios. These studies can be used to validate my formulae and I therefore cite examples below, but I share, and stress, Bollen's view [13] that practical investment management requires forward-looking, ex-ante, measures that account for the entire distribution of possible future payoffs.

Israelov and Klein[14] analyse collars and, as I do, emphasise the difference between cost and expected return. In their empirical study of collars they found a negative excess return, as I found in my example when using the implied volatility skew as of 3$^{\text{rd}}$ December 2021 implied volatility data. They measured the historical risk of a -5%/+10% moneyness collar to be 10.7% (with an underlying asset risk of 15.7%) whereas my formulae suggested it should, ex-ante, be 8.6%.

Slivka[15] measured the return and risk of a multitude of option strategies. He assumed a zero risk-premium for the underlying asset and therefore I focus solely on his risk estimates. His at-the money covered-call risk is 50% of the underlying, whereas mine is 45%. His at-the-money put-protection is 67% of the underlying, mine is 70%. His 5% moneyness call and put risk match mine exactly. However, his -5% moneyness call risk is 70% of underlying while mine is 60% and his -5% moneyness put is 45% of underlying whilst mine is 57%. It appears that we differ

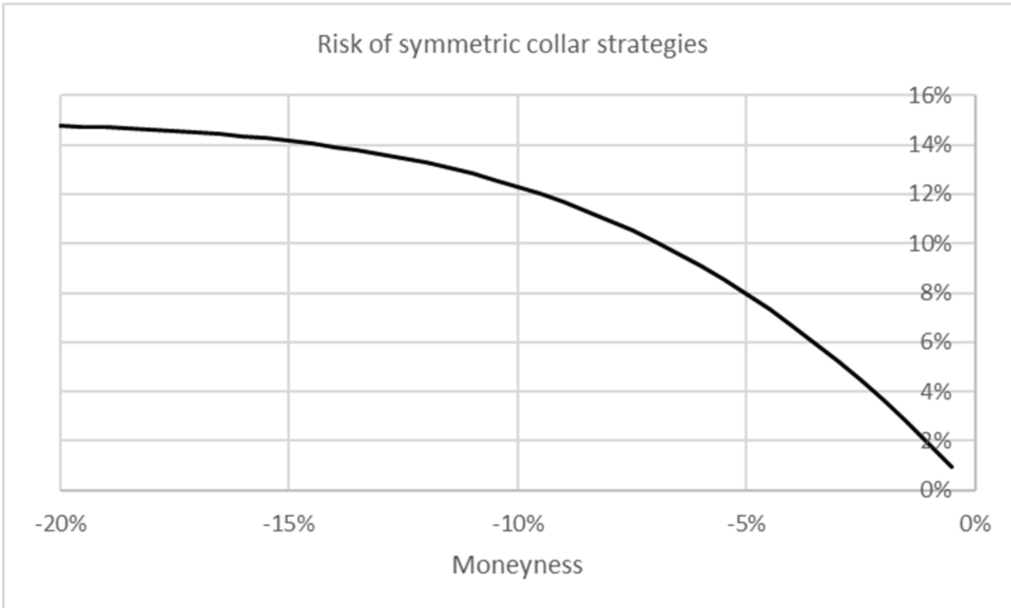

Figure 16. The risk of the return of an underlying asset with an overlaid symmetric collar strategy for a range of strike prices.

more for lower strike prices. Nevertheless his, and my, profiles of risk as strike changes in both strategies match.

Merton, Scholes and Gladstein[16] simulated call overwriting with various strike prices on historic stock data. Their results also match the profiles seen in my example. Both risk-premium and risk increase as the strike price increases. The authors' historic statistics are similar to my ex-ante formulae: for call-overwriting with -10%, 0%, +10%, and +20% moneyness, the authors find strategy returns of 3.3%, 3.7%, 4.5% and 5.3% while I get 3.9%, 4.6%, 5.8% and 6.8%. The authors find strategy risks of 4.9%, 7.1%, 9.3%, 11.2%, 16.6% while I get 2.9%, 6.0%, 9.5%, and 12.6%.

The aforementioned authors repeated their study for put-protection in[17]. Their returns for -10%, 0%, and +10% moneyness were 7.3%, 6.7%, and 5.9% while I got 7.1%, 6.2%, and 5.1%. Their risks were 12.0%, 9,5%, 7.1% while I got 14.9%, 12.6% and 9.4%. Again the profile of falling risk-premium and risk as moneyness increases is seen in this empirical study.

The nearest conceptual work to this article in the published literature, is the body of articles by Bookstaber and Clarke who focus on the distribution of returns of option adjusted portfolios, as laid out in [18] . In [19] they provided an algorithm for computing the return distribution of an option adjusted portfolio. My logic is very similar to theirs, but whereas Bookstaber and Clarke only provide a generic algorithm I have provided closed form formulae for the key summary statistics. In [20], the Bookstaber and Clarke apply their algorithm to simple call and put strategies and find the same profile in risk-premium and risk as anticipated by my formulae. I concur with their claim that this is due to a reduced exposure to the risky asset, as strike price changes, which would also imply a reduced beta which

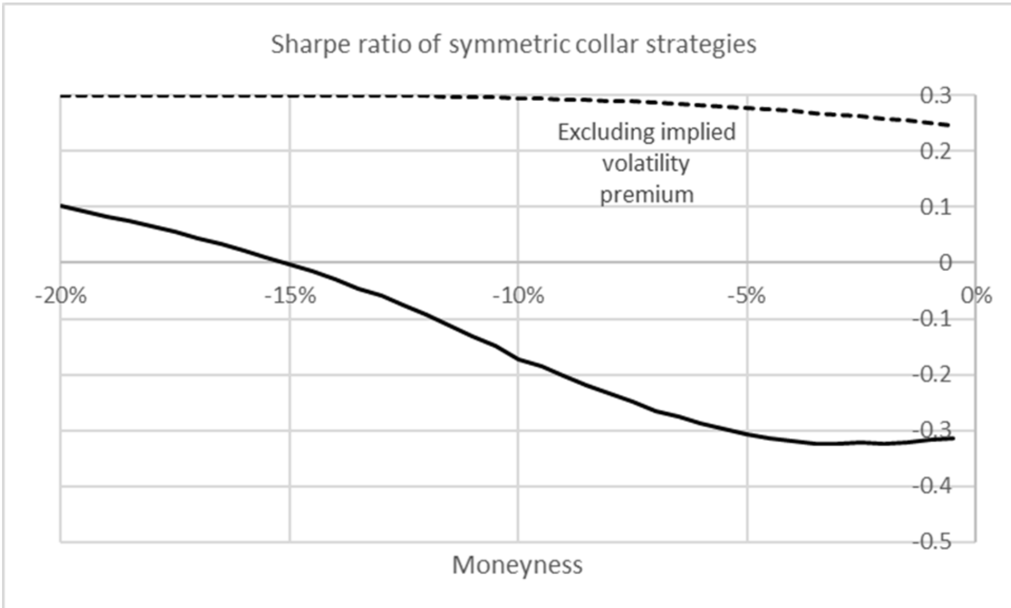

FIGURE 17. The Sharpe ratio for an underlying asset with an overlaid symmetric collar strategy for a range of strike prices. The dotted line assumes the implied volatility is equal to the underlying asset volatility. The solid line uses includes the implied volatility premium in the price of the options.

my formula also anticipates. Bookstaber and Clarke also observe that the option price bears no influence on the return distribution other than its location (i.e. the expected return) which is a characteristic evident in my formulae. My single-period histograms later in this article are similar to those in [21], however I extent the analysis to the more prevalent environment of single-period option investing being repeated over time. The authors caution against the use of measures such as standard-deviation and Sharpe ratio owing to the asymmetry that options introduce to a portfolio return distribution. This is an issue I will address in the next section.

6. Long-run return distribution

Thus far I have analysed option adjusted portfolios over a single investment period. The tenure of an option is typically a quarter-year, or sometime annual. The investment horizon of many investors, especially institutional investors such as pension funds, is significantly longer. Horizons of 3, 5 or even 10 or more years are commonplace. I now consider the behaviour of option adjusted portfolios over this longer period and shall use the term *long-run return* to represent the investment return of a portfolio over the investment horizon.

If the investment strategy includes options, then those options must be repurchased once they have expired, on a rolling basis, until the end of the investment horizon. The purpose of the second half of the paper is to identify the distribution of the long-run return of an option adjusted portfolio, assuming the rolling purchasing of options.

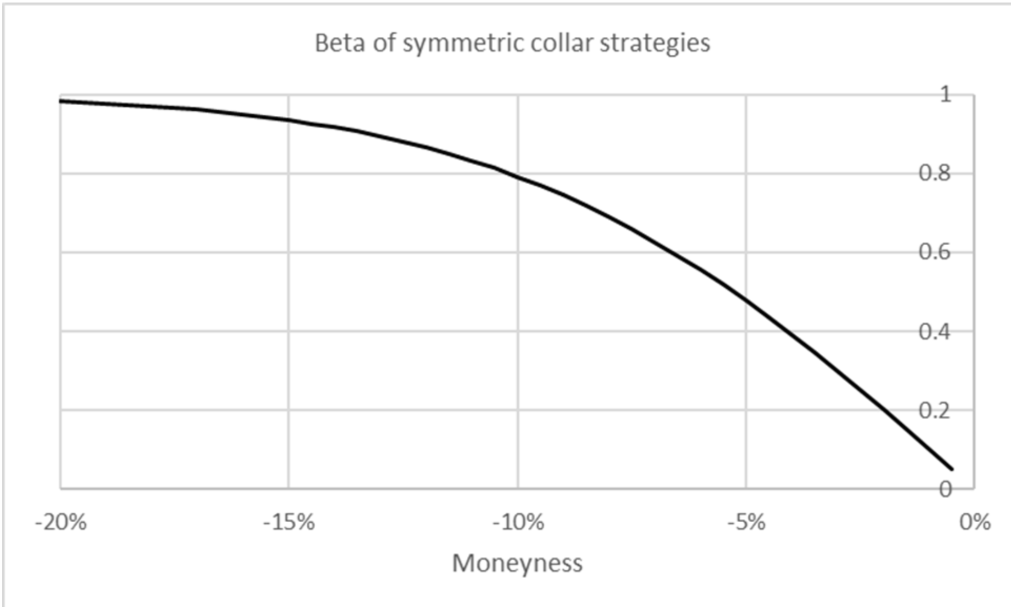

FIGURE 18. The price sensitivity (beta) of an underlying asset with an overlaid symmetric collar strategy for a range of strike prices, in relation to price change of the underlying asset.

If a new portfolio of options must be purchased when an existing portfolio of options has expired, the payoff profile could differ from period to period. I wish to consider the same strategy being sequentially deployed, and therefore define an option adjusted portfolio to be *invariant* if any two single-period option strategies are the same. That is, they have the same initial amounts in invested in the risk-free asset (as proportions of the starting underlying asset prices), the same number of units invested in the underlying asset, the same number of options, and every pair of corresponding options in the two strategies has the same moneyness.

For example, if the strike of a put-protection strategy is 80 when the asset price is 100, then, if when the strategy expires the asset price has risen to 120, the strike of the next put-protection strategy must be 96 (80% of 120) for the put-protection portfolio to be invariant.

Looking at the long-run performance of an investment strategy is problematic if the portfolio value becomes negative at any point. Fortunately, traditional financial assets, such as equities and bonds, can never have a negative price. However once options are introduced it is possible to construct option adjusted portfolios whose value at expiry can be negative. Such portfolios are said to be *leveraged*. Even if the portfolio value cannot be negative, it is possible to create a portfolio that has zero value even if the underlying asset price is not zero. Such strategies cannot be considered over the long-run and are therefore outside the context of this section of the article. When referring to option adjusted portfolios in this section, I will restrict my comments to *unleveraged*, *well-behaved*, portfolios; defined as portfolios whose value cannot be negative and can only be zero when the underlying asset value is zero.

I shall partition the investment horizon into m time segments, with the length of each segment being the tenure of the options. Therefore, the annualised long-run return of any investment strategy, denoted by R, is related to the annualised single unit returns, denote by $r_{(1)}, \ldots, r_{(m)}$, thus

$$1 + R = \left((1 + r(1)) \times \ldots \times (1 + r(m))\right)^{1/m} \tag{6.1}$$

which is logarithmically expressed as a simple average

$$\log(1 + R) = \frac{\log(1 + r_{(1)}) + \ldots + \log(1 + r_{(m)})}{m} \tag{6.2}$$

The conventional assumption is that asset prices follow a Geometric Brownian motion process over the investment horizon. In that case for an investment in that asset, $\log(1+r_{(1)}), \ldots \log(1+r_{(m)})$ is each normally distributed. The properties of the normal distribution tell us that the average of those log returns, $\log(1 + R)$, is also normally distributed. This fundamental result allows the analysis of an asset return over any investment horizon.

However, when options enter the portfolio the single time unit returns are no longer normally distributed. Indeed, the very purpose of introducing options is often to remove the symmetry of the return distribution. Fortunately, under certain conditions, we can state the long-run return distribution of option adjusted portfolios as m gets large. Those conditions are

(1) The underlying asset single-period returns are mutually independent and identically distributed

(2) The expectation and variance of the underlying asset single-period returns exist (i.e. are of finite size)

(3) The risk-free rate is unchanged through the investment horizon

(4) The option adjusted portfolio is invariant

The first three conditions are standard (see [22] and [23] for example) and warrant no further discussion other than stating that I assume them to hold in this analysis.

If these four conditions hold, I show in the Appendix that the long-run log return of these option adjusted portfolios will become normally distributed as m becomes large. The risk and expected return of the portfolio will of course depend on the payoff profile of the option strategy, but the long-run return will become normally distributed for *any* invariant option adjusted portfolio.

This is a crucial result. It holds for any distribution of underlying asset return (not just the log normal return of geometric Brownian motion) provided the first two conditions are met. It holds for every payoff profile. Regardless of how asymmetric the single period returns are, the long-run return distribution is symmetric. It will exhibit the same risk and symmetry as some combination of the underlying and the risk-free asset but can have a lower Sharpe ratio.

The speed of convergence to a normal distribution depends on the amount of asymmetry of single period returns. Nevertheless, as we will see from the following simulated examples the convergence is usually fast. Typically, symmetry of return is achieved within two years for strategies using quarterly options.

7. Examples: long-run

I will demonstrate the convergence of the long-run returns of option adjusted portfolios to a normal distribution with some simulations. First, I will simulate the single-period returns of the underlying asset (using the same summary statistics as specified in Section 3.4 above), then compute the single-period return for some option adjusted portfolios, and then compute annualised long-run returns of these portfolios. I will plot the histogram of the simulated long-run returns. In all cases I will set the ratio of the strike prices to the starting underlying asset price equal over time so that the option adjusted portfolios are invariant.

The speed of the convergence for each option strategy will be seen by plotting these histograms over time horizons of: one quarter, half year, three quarters, one year, five quarters, one and half years, seven quarters, two years, three years and four years.

Each graph will show histograms of the long-run returns of the option adjusted portfolios including, and excluding, the implied volatility premium. For reference, the histogram of the traditional portfolio consisting solely of the underlying asset and the risk-free asset, weighted to match the risk of the option adjusted portfolio, is also included.

7.1. **Call-overwriting.** First, I analyse a call-overwriting strategy, which is struck at-the-money. The extreme asymmetry created by the cap to the appreciation is clearly visible in the first panel of Figure 19. Also, the benefit of the receipt of the implied volatility risk premium can be seen by observing the distance between the spikes in the distribution including and excluding the implied volatility premium.

After four investment periods, the spike only occurs when all four quarterly asset prices exceeded the strike price. Therefore, a mode to the left of the spike begins to appear. After another four investment periods the spike has virtually disappeared, and a meaningful amount of symmetry has occurred. After three years the distribution is virtually symmetric and, although when excluding the implied volatility premium, the distribution misses some upside probability of a similarly risked asset portfolio and has higher probability of downside, the receipt of the implied volatility premium vastly offsets these small differences.

Figure 20 shows the set of histograms for a call-overwriting strategy that is struck 5% out-of-the-money. Whilst the distribution behaviour is similar to the at-the-money strategy, the convergence to normality occurs much more quickly, appearing symmetric after only six investment periods. This exemplifies the characteristic that the rate of convergence depends on the strike price: the less that the underlying asset return distribution is truncated by the strike price, the faster the convergence.

7.2. **Put-protection.** First, I analyse a put-protection strategy struck at-the-money. The simulation results are shown in Figure 21. The spikes, both when including and excluding the implied volatility premium, are to the left of the origin in the first panel. This represents that although the portfolio is protected from any loss, that protection has cost. The implied volatility premium itself simply causes the spike in returns distribution to be further left than the spike in returns when excluding the implied volatility premium.

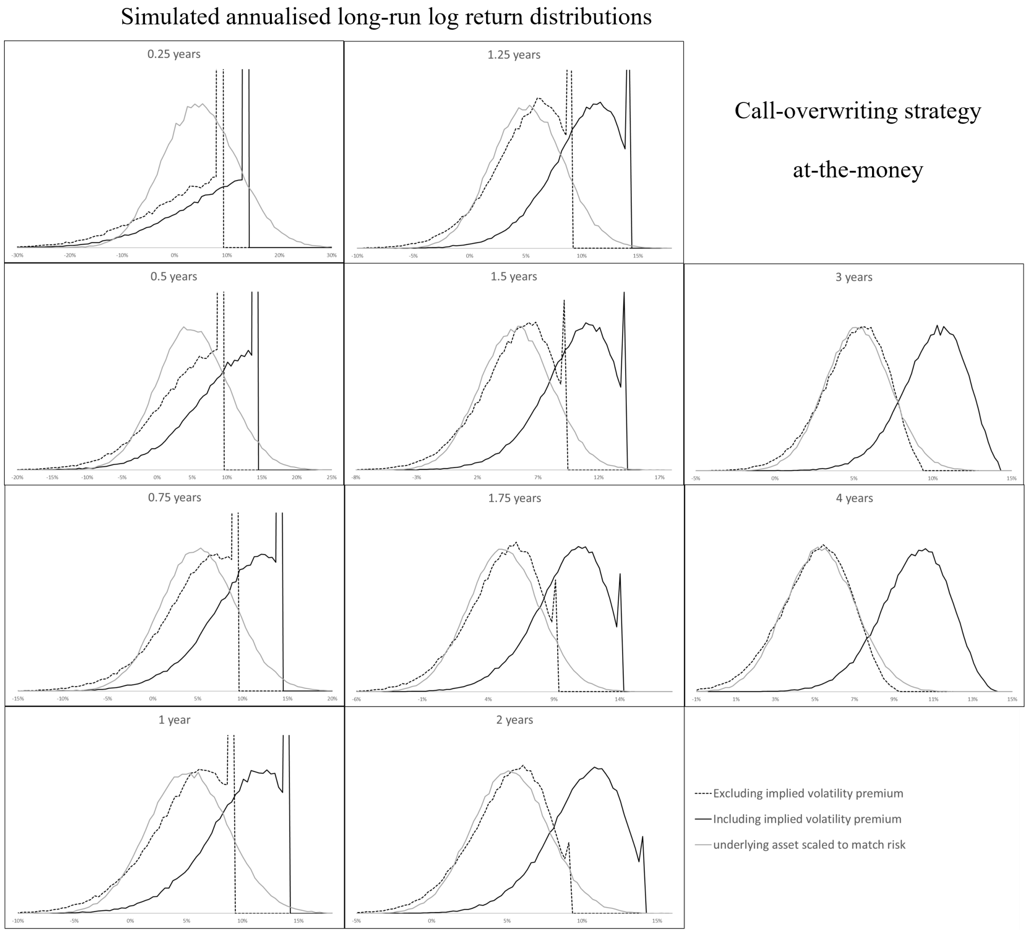

Figure 19. Simulations of the long-run returns of an underlying asset overlaid with a quarterly at-the money call-overwriting strategy, over investment horizons of 1/4 year, 1/2 year. 3/4 year, 1 year 1 1/4 years, 1 1/2 years, 1 3/4 years, 2 years, 3 years, and 4 years. The dotted line assumes the implied volatility is equal to the underlying asset volatility. The solid line uses includes the implied volatility premium in the price of the options. The grey line shows the long-run returns of a portfolio consisting of the risk-free asset and the underlying asset, weighted to have the same risk as the option strategy.

Like in the call-overwriting, after four investment-periods the spike in the return distribution only occurs if the underlying asset price exceeded the strike price on all four investment periods. Therefore, a mode, to the right of the spike, begins to form. Within another four investment periods symmetry is achieved. However, the cost of the implied volatility risk premium is clearly visible and if investors can toleration losses in a two-year period it may be better not to protect a portfolio

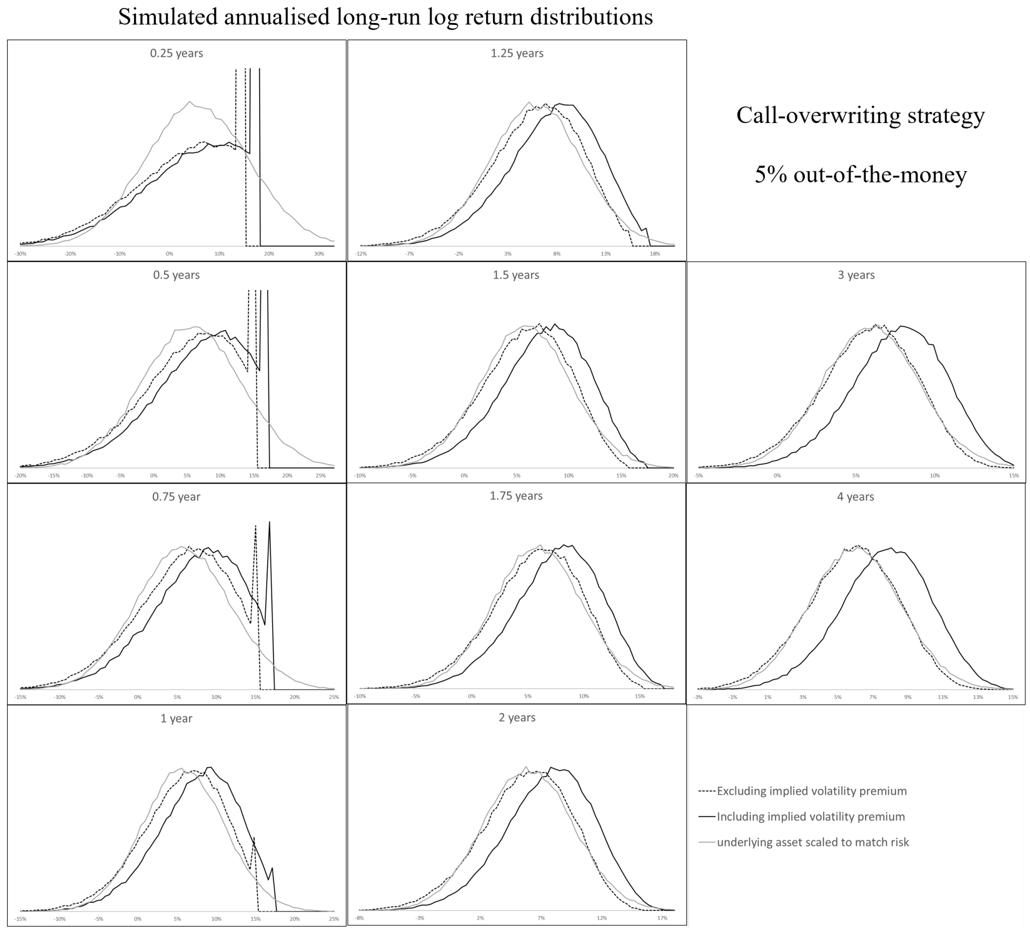

FIGURE 20. Simulations of the long-run returns of an underlying asset overlaid with a quarterly 5% out-of-the money call-overwriting strategy, over investment horizons of $1/4$ year, $1/2$ year. $3/4$ year, 1 year $1 1/4$ years, $1 1/2$ years, $1 3/4$ years, 2 years, 3 years, and 4 years. The dotted line assumes the implied volatility is equal to the underlying asset volatility. The solid line uses includes the implied volatility premium in the price of the options. The grey line shows the long-run returns of a portfolio consisting of the risk-free asset and the underlying asset, weighted to have the same risk as the option strategy.

with a put. Oddly, even when excluding the implied volatility premium, the put-protected portfolio does not converge exactly to a similarly risk scaled portfolio of the underlying asset and the risk-free asset. I do not have an explanation for that small difference in the location of these two distributions.

When the put-protection level is lowered to 5% out of the money, Figure 22 shows that the convergence to normality occurs more quickly, with the distribution being virtually symmetric after just one year. The effect of volatility skew can be

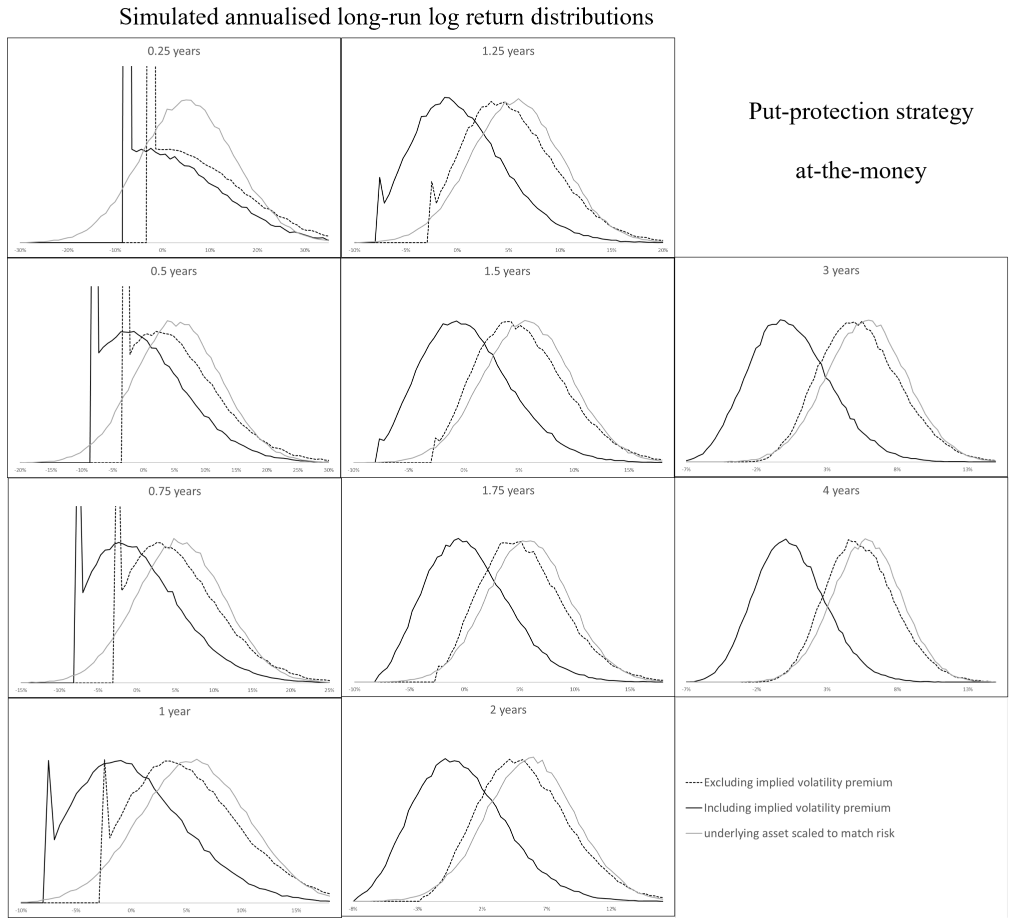

Figure 21. Simulations of the long-run returns of an underlying asset overlaid with a quarterly at-the money put-protection strategy, over investment horizons of 1/4 year, 1/2 year. 3/4 year, 1 year 1 1/4 years, 1 1/2 years, 1 3/4 years, 2 years, 3 years, and 4 years. The dotted line assumes the implied volatility is equal to the underlying asset volatility. The solid line uses includes the implied volatility premium in the price of the options. The grey line shows the long-run returns of a portfolio consisting of the risk-free asset and the underlying asset, weighted to have the same risk as the option strategy.

seen by observing the distance between the spikes when including and excluding the implied volatility premium in the 4-year panel and comparing that to the same distance of the 4-year panel of Figure 21. I also note that the unusual difference in the location of the dotted and grey distributions in Figure 21 is no longer present in Figure 22.

7.3. **Collars.** The simulations of a symmetric collar stuck at $\pm 5\%$ are shown in Figure 23. In the first panel, the two spikes at the cap and protection levels can

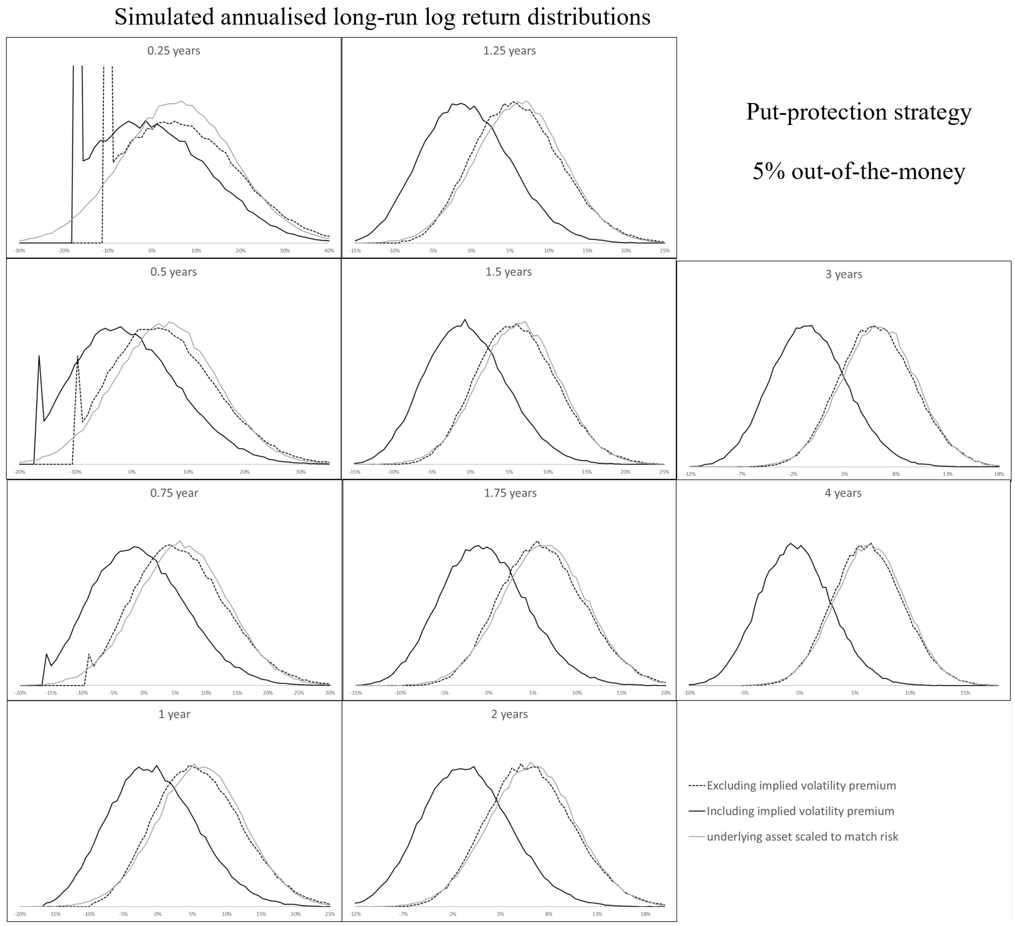

FIGURE 22. Simulations of the long-run returns of an underlying asset overlaid with a quarterly 5% out-of-the-money call-overwriting strategy, over investment horizons of $^1/_4$ year, $^1/_2$ year. $^3/_4$ year, 1 year $1^1/_4$ years, $1^1/_2$ years, $1^3/_4$ years, 2 years, 3 years, and 4 years. The dotted line assumes the implied volatility is equal to the underlying asset volatility. The solid line uses includes the implied volatility premium in the price of the options. The grey line shows the long-run returns of a portfolio consisting of the risk-free asset and the underlying asset, weighted to have the same risk as the option strategy.

be observed. After the second quarter there are three spikes: the protection being triggered in both investment periods, the cap being triggered in both periods, or the protection being triggered in one period and the cap being triggered in the other. At all time-horizons, the distributions are symmetric but not normal, until after two years. Again, the impact of the implied volatility premium can be seen. Even though the cost of the option strategy has been reduced by selling the call, the overall return distribution quickly appears normal but with lower mode than a suitably risk scaled portfolio of the underlying asset and the risk-free asset.

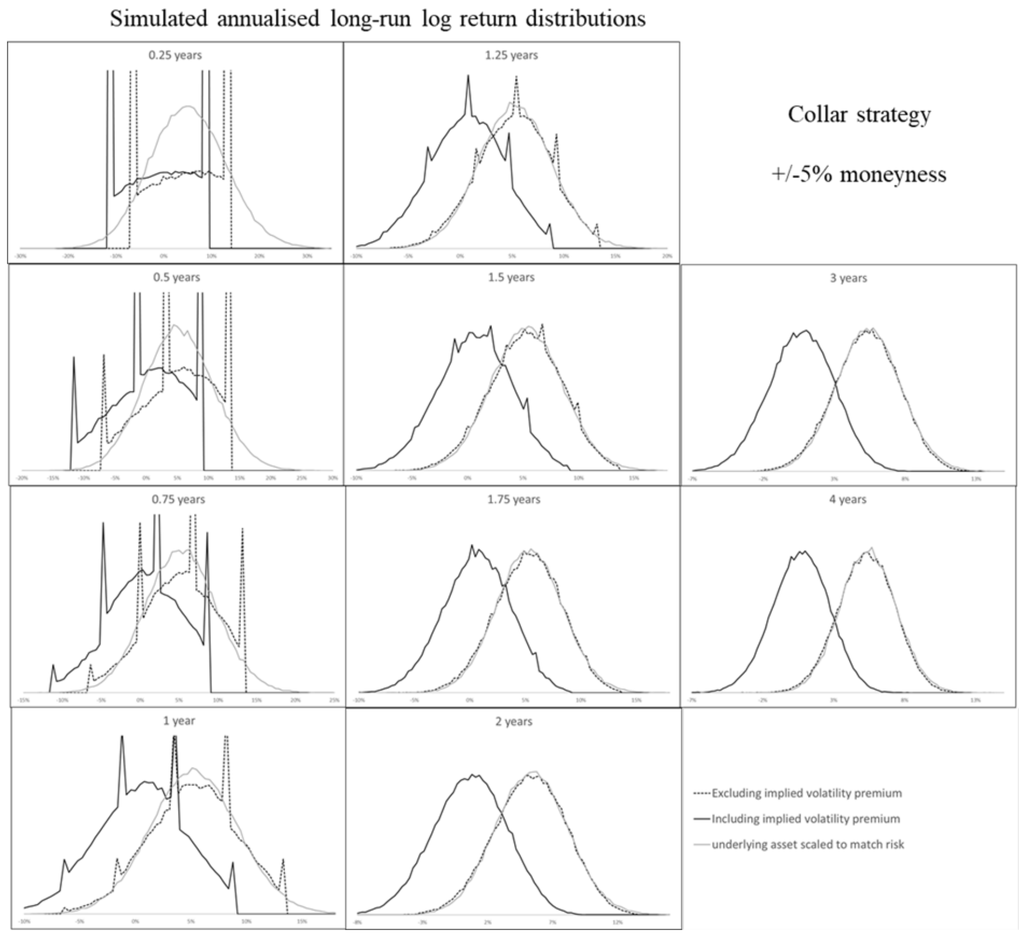

FIGURE 23. Simulations of the long-run returns of an underlying asset overlaid with a quarterly ±5% moneyness symmetric collar strategy, over investment horizons of 1/4 year, 1/2 year. 3/4 year, 1 year 1 1/4 years, 1 1/2 years, 1 3/4 years, 2 years, 3 years, and 4 years. The dotted line assumes the implied volatility is equal to the underlying asset volatility. The solid line uses includes the implied volatility premium in the price of the options. The grey line shows the long-run returns of a portfolio consisting of the risk-free asset and the underlying asset, weighted to have the same risk as the option strategy.

Figure 24 shows the distributions from a collar strategy where the strikes are now ±10% from current price of the underlying asset. The convergence behaviour is consistent with that observed in Figure 23 but occurs much faster. In this case full convergence to a normal distribution is achieved in one year.

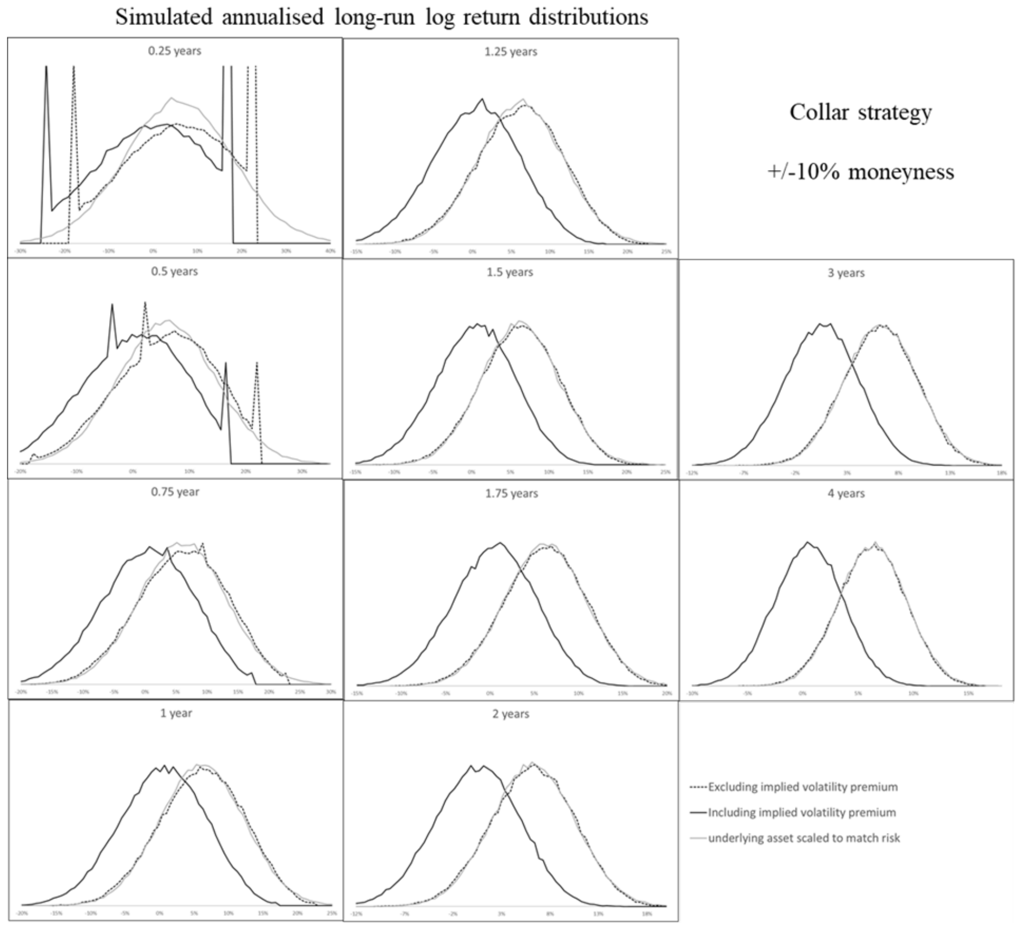

Figure 24. Simulations of the long-run returns of an underlying asset overlaid with a quarterly ±10% moneyness symmetric collar strategy, over investment horizons of 1/4 year, 1/2 year. 3/4 year, 1 year 1 1/4 years, 1 1/2 years, 1 3/4 years, 2 years, 3 years, and 4 years. The dotted line assumes the implied volatility is equal to the underlying asset volatility. The solid line uses includes the implied volatility premium in the price of the options. The grey line shows the long-run returns of a portfolio consisting of the risk-free asset and the underlying asset, weighted to have the same risk as the option strategy.

8. Assumptions and Caveats

I have made several assumptions on the characteristics of the underlying asset returns in this article, and whilst they are considered as standard assumptions (see [2] for example), a couple of them do nevertheless have an impact on my results. I therefore briefly discuss them here.

Over a single-period, other than bearing some default risk, risk-free securities do exist in practice. However, the risk-free rate of return does change over the long-run. The main impact of this change is on the price of the option, which will then change over time, even for invariant strategies. The convergence of long-run return to normality is affected by this change because $\mathbb{E}[r_O]$, will differ over time. Fortunately, the changes in risk-free rate are so small relative to the underlying asset volatility that this does not have a large impact. The risk-free rate for borrowing differs in practice to the risk-free rate for lending. This causes r_O to differ for overlays on asset portfolios relative to option strategies on a risk-free investment. Applying strategies as overlays are much more common in practice, and the options purchased from a cash buffer. Therefore, it is not common for the investor to physically borrow to implement an option strategy and the lending risk-free rate should be used in my formulae.

It is theoretically not possible to buy and sell securities in any amount. Physical short selling is not permitted in some jurisdictions, for example, and credit lines will not be given to create unlimited leverage. For the common option strategies, including my three examples, the exposure to the risky asset will never drop below 0 or exceed the value of the investment endowment. This assumption is therefore innocuous.

Although geometric Brownian motion is a widely adopted assumption for asset prices, it has an impact on my results, because it governs, for example, the tail behaviour of the underlying asset returns. The fact that implied volatility increases as the moneyness of options increases suggest the market believes that fatter tails may prevail. Fortunately, my formulae are easy to adjust if another price process is preferred (although the pricing of options using other processes is outside the scope of this article).

It is possible that the distribution of r_A is so fat tailed that its variance and expected return do not exist; for example if r_A follows a stable distribution[24]. The convergence to normality of long-run returns would fail in this case, although the long run return would still be symmetric. However, if $\mathbb{E}[r_A]$ did not exist then the price of an option would also be infinitely big. The fact that options have finite prices implies that the expectation of r_A exists. Therefore, I believe the existence of $\mathbb{E}[r_A]$ and $\mathbb{V}[r_A]$ to be reasonable assumptions.

In reality the underlying expected return and variance are not known. However, our objective is forming summary statiscs for option adjusted portfolios *condition on* $\mathbb{E}[r_A]$ and $\mathbb{V}[r_A]$. Therefore this issue falls outside the scope of this article, but estimation error remains a broader problem error even for traditional portfolios.

The efficient market hypothesis argues for independent returns[25] and empirically, asset returns exhibit near independence for all except the lowest risk assets. However, some equity price momentum has been observed[26]. Even if there is some dependence, generalisations of the central limit theorem state that the long-run return still converges to a normal distribution - albeit at a potentially slower rate. In practice, serial correlation does mean that the standard error of estimates of the summary statistics computed from sample data can be greater than expected.

Expected returns and volatilities are unlikely to be fixed over time. Heteroskedasticity has long been observed in financial return data[27] and, in part, explains the implied volatility premium. This volatility of volatility means that the long-term options strategies do not necessarily become normally distributed. However, the tending toward a normal distribution can still occur for moderate differences in distribution

The no arbitrage assumption needs no further comment as it underpins pricing theory and is not expected to be violated.

All the risk of violations of the above assumptions are embedded in the implied volatility premium. Therefore, when simulating under the assumptions above, by including the implied volatility premium I have introduced a "free-lunch" that should not be expected to appear in practice. This means that the put-protection and collar strategy distributions will probably not be as distant from the distribution of the underlying asset in reality. It also means the premium harvested in the call-overwriting strategies is probably overstated in my results. Given I have simulated the underlying asset returns using the above assumptions, the dotted lines in my figures showing the risk-premium excluding the implied volatility premium are a better comparator to the underlying asset.

9. Conclusions

Under certain definitions of portfolio efficiency, options can play a useful role in investment and are seeing an increased use. For example, they can be used to enhance the income of a portfolio and to protect the downside when more risk is prevalent in the portfolio than the investor is comfortable with. However, the impact of the inclusion of option strategies on the traditional measures of portfolio efficiency, such as expected return, risk, and beta, should be assessed.

I have shown in this article that closed form expressions exist for the risk-premium, risk, and beta of any option strategy; and further showed that these expressions simplify into straight forward calculations for the common strategies such as call-overwriting, put-protection, and collars.

Furthermore, I showed that these summary statistics are materially impacted by options, but in an intuitive way, with the positive relationship between risk and premium being largely preserved. With more return outcomes being truncated by the option strategies, the more the portfolio behaves like a risk-free asset and the more the risk-premium and beta are reduced accordingly. My objective was to illustrate that options should not be overlaid on existing portfolio in order to engineer certain outcomes such as enhanced income whilst assuming that the risk, risk-premium, and beta of the portfolio are unaffected.

I have also shown that repeatedly deploying an option strategy will result, quite quickly, in the portfolio long-run return being normally distributed, regardless of the amount of asymmetry imposed on the portfolio by the options. Therefore, the investor should carefully elicit the time horizon over which the engineered outcome from the options is required. It is clear from the simulations that the payoff profile over a single time-period, is not preserved over multiple time-periods. An investment strategy that is efficient over a single period may not be efficient in the long run. It may be that a simple combination of the underlying risky asset and the risk-free asset yields suitable long-term return characteristics.

Appendix A

The purpose of this Appendix is to detail the two key results in this article: i) deriving formulae for the risk-premium, risk, and beta of an option adjusted portfolio, and ii) proving that the long-term return of an option adjusted portfolio.

To meet these objectives, I first show that put and call options are related (both having finite price) and therefore show that every option adjusted portfolio can be considered as if the option component consists only of call options. I proceed by defining a continuous piecewise linear function and show that a portfolio containing call options has a value function of this form. With this I derive the mean, variance, and covariance, of the portfolio value function at expiry, which I prove are finite quantities. To meet my first objective, I use this result to express formulae for the risk-premium, risk, and beta of a portfolio including options.

I then define an identical option strategy and show that a time series of returns from identical option strategies are identically distributed, and that they are independent. Since the mean and variance exist, I show that the annualised long-run return must tend to a normal distribution.

Some of the properties in this Appendix are well-known. However, I have not seen proofs of all of them, certainly not in a single document, and I therefore prove all properties stated here to form a self-contained body of work and to enable me to develop notation an concepts used throughout the Appendix.

Definition A.1. An *asset* is an item with value which can be bought or sold at a price [28].

Throughout this Appendix I will be considering the price change over a specific period of time, and thus will denote the asset price at start and end of the time period as s and x respectively. s is strictly positive, and x is non-negative but could be zero valued. Conceptually, the start of the time-period is in the past and the end is in the future so that s is known but x is a random variable.

Definition A.2. The *risk-free asset* is an asset whose end point price is known at the start point[28]. An asset that is not risk-free is termed a *risky* asset.

I will denote the risk-free asset's start and end price by s_F and x_F respectively. Both s_F and x_F are strictly positive.

Assumption A.3. A risky asset and a risk-free asset are available to be bought and sold in any amount, and without trading frictions.

Definition A.4. A *call option* is the right, but not the obligation, to buy a unit of the risky asset on a specified date, termed the *expiry date* of the option, at a price agreed at the time of origination of the option, termed the *strike price*. A *put option* is the right, but not the obligation, to sell a unit of the risky asset on a specified date at a specified price.[1]

In this Appendix all options have an expiry date equal to the end point of the investment time period.

Definition A.5. A *portfolio* is a collection of assets and options[28]. An element of the portfolio is called a *constituent*.

Multiple options can appear in a single portfolio, all potentially having different strike prices, but all with the same expiry date. I will be specific about what is held in a portfolio, using statements such as 'a portfolio consisting of the risk-free asset and a call option' and, where I need to be explicit, I will use the term *option adjusted portfolio* to refer to a portfolio that contain options.

Definition A.6. A *value function*, denoted in this Appendix by $v(.)$, is the value of an asset, or option, or portfolio, at the end of the time-period, as a function of the underlying asset price, x.

It is commonly referred to as the *payoff profile*[4] but in this Appendix I will use the term *value function* to reinforce the fact that it is a function of the asset price.

Property A.7 (Value of assets and options). The value functions of various assets are as follows:
For the risk-free asset, $v(x) = x_F$[4]
For an asset, $v(x) = x$[4]
For a call option, $v(x) = \max(x - k, 0)$ where k denotes the strike price[4]
For a put option, $v(x) = \max(k - x, 0)$ where k denotes the strike price[1]
For an option adjusted portfolio consisting of n_p put options, with q_i units of the ith put option which has strike price k_i $(i = 1, \ldots, n_p)$, n_c call options, with q_{i+n_p} units of the ith call option with strike price $k_{i+n_p}(i = 1, \ldots, n_c)$, q_A units of the underlying asset, and q_F invested in the risk-free asset, the value function is

$$v(x) = q_F x_F + q_A x + \sum_{i=1}^{n_p} q_i \max(k_i - x, 0) + \sum_{i=n_p+1}^{n_c} q_i \max(x - k_i, 0)$$

Proof. By definition, the risk-free asset has a constant value at the end of the investment period and is not dependent on the risky asset. Therefore $v(x) = x_F$ for the risk-free asset.
The value of an investment in the risky asset is its price. Therefore $v(x) = x$ for the risky asset.
If the risky asset price, x, is lower than the strike price, k, of a call option then the option will not be exercised and expires worthless. Otherwise, the option is exercised, and the option owner can buy the risky asset at price k and immediately sell it at its price of x. Therefore $v(x) = \max(x - k, 0)$ for a call option.
Similarly, if the risky asset price is greater than the strike price of a put option, then the option expires worthless. Otherwise, the option is exercised, and the option owner can sell the risky asset at price k and immediately buy it back at price x. Therefore $v(x) = \max(k - x, 0)$ for a put option.

The value of a portfolio is the sum of the value of its constituents. The value of each constituent is the value of a single unit of each constituent multiplied by the number of units owned. Therefore, the portfolio value function is

$$v(x) = q_F x_F + q_A x + \sum_{i=1}^{n_p} q_i \max(k_i - x, 0) + \sum_{i=n_p+1}^{n_c} q_i \max(x - k_i, 0)$$

□

Property A.8 (Put-call parity). A portfolio consisting of an underlying asset and a put option on that asset has the same value function as a portfolio consisting of a call option and an amount invested in the risk-free asset to buy the underlying asset if it were called[1].

Proof. A portfolio consisting of an underlying asset and a put option on that asset has a value function of $\max(k-x, 0)+x$, where k denotes the strike price of the put and x is the underlying asset price at expiry. This can be expressed as $\max(k, x)$.

The amount needed to buy the underlying asset at the strike price on the expiry date, must be k. Therefore a portfolio consisting of a call option and an amount invested in the risk-free asset to buy the underlying asset if it were called has a value function of $\max(x-k,0)+k$. This also can be expressed as $\max(k,x)$. □

Definition A.9. An *arbitrage* exists if a portfolio with zero price has a non-negative value function (and the value function is strictly positive somewhere)[28].

Assumption A.10. There are no arbitrage opportunities.

Property A.11 (Existence of option price). The price of both put and call options exist[29].

Proof. I focus on a put option first. I will show that its price is bounded above and below and therefore, must exist.

No arbitrage ensures that there cannot exist a portfolio with zero starting value whose value function at expiry is positive for every price of the underlying asset.

Consider the portfolio consisting of a put option and a short position in the risk-free asset equal in size to the put option price. This portfolio has zero starting value. Its value function at expiry is $v(x) = \max(k-x,0) - p_{put}\frac{x_F}{s_F}$. where p_{put} denotes the price of the put option. If $p_{put} < 0$ then $v(x) > 0$ for every value of x, which breaches the no arbitrage condition.

Consider now the reverse of this portfolio: a long position in the risk-free asset equal in size to the put option price, and short a put option. Its value function is $v(x) = p_{put}\frac{x_F}{s_F} - \max(k-x,0)$. If $p_{put} > k/\frac{x_F}{s_F}$ then $v(x) > 0$ for every value of x, which also breaches the no arbitrage condition. Therefore, p_{put} is bounded and so must exist.

I now prove that a call option price exists. From the put-call parity property, the expiry value of a put option and the underlying asset equals the value of a call option and $k/\frac{x_F}{s_F}$ invested in the risk-free asset. Therefore $p_{put} + s = p_{call} + k/\frac{x_F}{s_F}$ where p_{call} denotes the price of the call option. Since s_F and x_F are finite quantities with x_F strictly positive, and since I have shown that p_{put} is bounded, then p_{call} is also bounded and therefore must also exist. □

Property A.12 (Sufficiency of call options). Every portfolio consisting of put and call options on an underlying asset, the underlying asset, and the risk-free asset can be expressed as a portfolio of solely call options, the underlying asset and the risk-free asset.

Proof. This proof is based on put-call parity. The absence of arbitrage means that any two portfolios with the same value function must have the same price. Therefore, I need only demonstrate that the value function of any portfolio containing put options is equal to the value function of another portfolio not containing put options.

Let us suppose that the portfolio consists of n_p put options, with q_i units of the ith put option which has strike price k_i $(i = 1, \ldots, n_p)$, n_c call options, with q_{i+n_p} units of the ith call option with strike price $k_{i+n_p}(i = 1, \ldots, n_c)$, q_A units of the underlying asset, and q_F units of the risk-free asset.. The portfolio value function is

$$v(x) = q_F x_F + q_A x + \sum_{i=1}^{n_p} q_i \max(k_i - x, 0) + \sum_{i=n_p+1}^{n_c} q_i \max(x - k_i, 0)$$

Now, $\max(k_i - x, 0) = \max(x - k_i, 0) - x + k_i$ and therefore we can write the value function as

$$v(x) = (q_F + \frac{1}{x_F}\sum_{i=1}^{n_p} q_i k_i)x_F + (q_A - \sum_{i=1}^{n_p} q_i \max(k_i - x, 0))x + \sum_{i=1}^{n_c} q_i \max(x - k_i, 0)$$

This is the value function of a portfolio consisting of solely call options, the underlying asset and the risk-free asset. □

Definition A.13. A function f is *piecewise linear* on $[0, \infty)$ if for some partition $0 = x_0 < x_1 < \ldots < x_m < x_{m+1} = \infty$, it can be expressed as

$$f(x) = a_i + b_i(x - x_i) \qquad \text{for } x_i \leq x < x_{i+1} \quad \text{and for } i = 0, \ldots, m$$

where $a_0, \ldots, a_m$ and $b_0, \ldots, b_m$ are finite constants[30]. (Note that the requirement that the constants are finite excludes vertical segments in the function).

Definition A.14. A function f is *continuous* at ξ if $\lim_{x \to \xi} f(x) = f(\xi)$, and continuous *everywhere* if it is continuous at every point on its domain[30].

Definition A.15. A *continuous piecewise linear function* is a piecewise linear function that is continuous everywhere on its domain.

Property A.16 (Defining the intercepts of a continuous piecewise linear function). A function f is a continuous piecewise linear function on $[0, \infty)$ if, and only if, it can be expressed as

$$f(x) = a_i + b_i(x - x_i) \qquad \text{for } x_i \leq x < x_{i+1} \quad \text{and for } i = 0, \ldots, m$$

where $a_i = a_{i-1} + b_{i-1}(x_i - x_{i-1})$ for $i = 1, \ldots, n$.

Proof. By definition, if a function f is piecewise linear on $[0, \infty)$ then it can be expressed as

$$f(x) = a_i + b_i(x - x_i) \qquad \text{for } x_i \leq x < x_{i+1} \quad \text{and for } i = 0, \ldots, m$$

If f is continuous everywhere, then in particular it is continuous at x_i for $i = 1, \ldots, m$. That implies that $\lim_{\epsilon \to 0} f(x_i - \epsilon) = f(x_i) = a_i$. Now,

$$\lim_{\epsilon \to 0} f(x_i - \epsilon) = \lim_{\epsilon \to 0} a_{i-1} + b_{i-1}(x_i - \epsilon - x_{i-1}) = a_{i-1} + b_{i-1}(x_i - x_{i-1})$$

Thus $a_i = a_{i-1} + b_{i-1}(x_i - x_{i-1})$ as required.

Conversely, if $f(x) = a_i + b_i(x - x_i)$ for $x_i \leq x < x_{i+1}$ and for $i = 0, \ldots, m$ where $a_i = a_{i-1} + b_{i-1}(x_i - x_{i-1})$, then $\lim_{\epsilon \to 0} f(x_i - \epsilon) = f(x_i)$. Since the function is piecewise linear, and the domain of each segment includes the lower point of its partition, then $\lim_{\epsilon \to 0} f(x_i + \epsilon) = f(x_i)$. Since $\lim_{\epsilon \to 0} f(x_i - \epsilon) = \lim_{\epsilon \to 0} f(x_i + \epsilon)$, then $\lim_{x \to x_i} f(x) = f(x_i)$ and therefore f is continuous at x_i for $i = 0, \ldots, m$. Furthermore, since all other points on the domain lie in the open interior of a line segment, f must be continuous at those points. Therefore, f is continuous everywhere as required. □

Property A.17 (Equivalence of the value of portfolios of options and continuous piecewise linear value functions). Every option adjusted portfolio has a continuous piecewise linear value function; and every continuous piecewise linear function is the value function of some option adjusted portfolio.

This property is a simple extension into the continuous domain of the well-known discrete case proven in [31].

Proof. Proving that every option adjusted portfolio has a continuous piecewise linear value function:

From the Sufficiency of Call Options Property, we can assume without loss of generality that the portfolio of options, the underlying asset, and the risk-free asset contains no puts, only calls. It therefore has a value function expressed as

$$v(x) = q_F x_F + q_A x + \sum_{i=1}^{n} q_i \max(x - x_i, 0)$$

where q_F denotes the number of units of the risk-free asset owned, q_A denotes the number of units of the underlying asset owned, n denotes the number of options, and q_i denotes the number of units of the ith call option owned with strike price x_i for $i = 1, \ldots, n$. I order the n options such that $x_1 < \ldots < x_n$ and add $x_0 = 0$ and $x_{n+1} = \infty$ enabling me to write that every $x \geq 0$ satisfies $x_i \leq x < x_{i+1}$ for some i between 0 and n. Therefore, I can state that when $x_i \leq x < x_{i+1}$

$$\sum_{i=1}^{n} q_i \max(x - x_i, 0) = \sum_{j=1}^{i} q_j (x - x_j)$$

This implies that when $x_i \leq x < x_{i+1}$

$$\begin{aligned} v(x) &= q_F x_F + q_A x + \sum_{j=1}^{i} q_j (x - x_j) \\ &= q_F x_F + q_A x_i + \sum_{j=1}^{i} q_j (x_i - x_j) + (q_A + \sum_{j=1}^{i} q_j)(x - x_i) \end{aligned}$$

By setting $a_i = q_F x_F + q_A x_i + \sum_{j=1}^{i} q_j (x_i - x_j)$ and $b_i = q_A + \sum_{j=1}^{i} q_j$ I can write that when $x_i \leq x < x_{i+1}$

$$v(x) = a_i + b_i (x - x_i)$$

and since this expression for $v(x)$ applies for x in every partition, we have shown that v is a piecewise linear function. Furthermore, for $i = 1, \ldots, n$

$$\begin{aligned} a_i &= q_F x_F + q_A x_i + \sum_{j=1}^{i} q_j (x_i - x_j) \\ &= a_{i-1} + (q_A + \sum_{j=1}^{i-1} q_j)(x_i - x_{i-1}) \\ &= a_{i-1} + b_{i-1} (x_i - x_{i-1}) \end{aligned}$$

This is the condition from the *Defining the intercepts of a continuous piecewise linear function* property, and therefore v is a continuous piecewise linear function.
Proving that every continuous piecewise linear function is the value function of some option adjusted portfolio:

By definition, if a function v is continuous and piecewise linear on $[0, \infty)$ then for some partition $0 = x_0 < x_1 < \ldots < x_n < x_{n+1} = \infty$, it can be expressed as

$$v(x) = a_i + b_i(x - x_i) \qquad \text{for } x_i \leq x < x_{i+1} \quad \text{and for } i = 0, \ldots, n$$

where $a_i = a_{i-1} + b_{i-1}(x_i - x_{i-1})$. From this recurrence relationship we can write that

$$a_i = a_0 + \sum_{j=0}^{i-1} b_j(x_{j+1} - x_j)$$

and so

$$\begin{aligned} v(x) &= a_0 + \sum_{j=0}^{i-1} b_j(x_{j+1} - x_j) + b_i(x - x_i) \\ &= a_0 - \sum_{j=0}^{i} b_j x_j + \sum_{j=0}^{i-1} b_j x_{j+1} + \sum_{j=0}^{i} b_j x - \sum_{j=0}^{i-1} b_j x \\ &= a_0 + b_0 x + \sum_{j=1}^{i} (b_j - b_{j-1})(x - x_j) \end{aligned}$$

But since $0 = x_0 < x_1 < \ldots < x_n < x_{n+1} = \infty$ we know that

$$\sum_{j=1}^{i} q_j(x - x_j) = \sum_{i=1}^{n} q_i \max(x - x_i, 0) \quad \text{for } x_i \leq x < x_{i+1}$$

which means that

$$v(x) = a_0 + b_0 x + \sum_{i=1}^{n} (b_i - b_{i-1}) \max(x - x_i, 0) \quad \text{for all } x \geq 0$$

This is the value function of an option adjusted portfolio containing $\frac{a_0}{/x_F}$ units of the risk-free asset, b_0 units of the underlying asset, and $(b_i - b_{i-1})$ units of a call option with strike price x_i, where $i = 1, \ldots, n$. □

Definition A.18. The *investment return* is the change in value of the investment, over the investment period, as a proportion of the starting value.[2]
The risk-free asset return is $r_F = \frac{x_F}{s_F} - 1$
The risky asset return is $r_A = \frac{x}{s} - 1$
The option adjusted portfolio return is $r_O = \frac{v(x)}{s} - \frac{x_F}{s_F}.\frac{p}{s} + \frac{x_F}{s_F} - 1$
where $v(x)$ is the portfolio value function derived from the constituents of the portfolio, and p is the purchase price of the portfolio.

The intuition behind the option adjusted portfolio return is that an investment is made in the risk-free asset, the price of the option portfolio is borrowed at the risk-free rate and used to purchase the option portfolio. Substituting the risk-free asset return allows us to rewrite r_O as

$$r_O = \frac{v(x)}{s} - (1 + r_F)\frac{p}{s} + r_F$$

The returns r_A and r_O are a function the random variable $\frac{x}{s}$, which I shall label q. I shall denote the probability density function of q by $g(.)$.

Property A.19 (Mean and variance of an asset return). The expectation and variance of the risky asset return, $\mathbb{E}[r_A]$ and $\mathbb{V}[r_A]$ respectively, (where $\mathbb{E}[.]$ denotes the expectation operator and $\mathbb{V}[.]$ denotes the variance operator), are

$$\mathbb{E}[r_A] = Y - 1 \qquad \text{and} \qquad \mathbb{V}[r_A] = Z - Y^2$$

where $Y = \int_0^\infty q\, g(q)dq$ and $Z = \int_0^\infty q^2\, g(q)dq$

Proof.
$\mathbb{E}[r_A] = \mathbb{E}[\frac{x}{s}] - 1 = Y - 1$ where $Y = \int_0^\infty q\, g(q)dq$
$\mathbb{V}[r_A] = \mathbb{E}[r_A^2] - \mathbb{E}[r_A]^2$
$\mathbb{E}[r_A]^2 = Y^2 - 2Y - 1$
$\mathbb{E}[r_A^2] = \mathbb{E}[(\frac{x}{s})^2] - 2\mathbb{E}[\frac{x}{s}] - 1 = Z - 2Y - 1$ where $Z = \int_0^\infty q^2\, g(q)dq$
Therefore $\mathbb{V}[r_A] = Z - 2Y - 1 - (Y^2 - 2Y - 1) = Z - Y^2$. □

Property A.20 (Mean, variance and covariance of an option adjusted portfolio return). The return of an option adjusted portfolio with a value function $v(x) = a_i + b_i(x - x_i)$ for $x_i \leq x < x_{i+1}$ and for $i = 0, \ldots, n$ has expectation, variance, and covariance with risky asset return, as follows:

$$\mathbb{E}[r_O] = A + B - (1 + r_F)\frac{p}{s} + r_F$$

where $A = \sum_{i=0}^n A_i$ and $B = \sum_{i=0}^n B_i$, with $A_i = y_i K_i$ and $B_i = b_i(J_i - q_i K_i)$.
The variables y_i and q_i are defined as $y_i = \frac{a_i}{s}$ and $q_i = \frac{x_i}{s}$ for $i = 0, \ldots, n$.
The variables K_i and J_i are defined as $K_i = \int_{q_i}^{q_{i+1}} g(q)dq$ and $J_i = \int_{q_i}^{q_{i+1}} q\, g(q)dq$, for $i = 0, \ldots, n$.

$$\mathbb{V}[r_O] = C + D + 2E - (A + B)^2$$

where $C = \sum_{i=0}^n C_i$, $D = \sum_{i=0}^n D_i$, and $E = \sum_{i=0}^n E_i$,
with $C_i = y_i^2 L_i$, $D_i = b_i^2(L_i - 2q_i J_i + q_i^2 K_i)$, and $E_i = y_i b_i(J_i - q_i K_i)$.

$$\mathbb{COV}[r_O, r_A] = F + G - Y(A + B)$$

where $F = \sum_{i=0}^n F_i$ and $G = \sum_{i=0}^n G_i$, with $F_i = y_i J_i$ and $G_i = b_i(L_i - q_i J_i)$.
The variable L_i is defined as $L_i = \int_{q_i}^{q_{i+1}} q^2\, g(q)dq$, for $i = 0, \ldots, n$.

Proof. From the value function we have that

$$\frac{v(x)}{s} = y_i + b_i(\frac{x}{s} - q_i) \qquad \text{for } q_i \leq \frac{x}{s} < q_{i+1} \quad \text{and for } i = 0, \ldots, n$$

where, for $i = 0, \ldots, n$, $y_i = \frac{a_i}{s}$ and $q_i = \frac{x_i}{s}$.

The first moment is written as

$$\begin{aligned}
\mathbb{E}\left[\frac{v(x)}{s}\right] &= \int_{q_0}^{q_1}(y_0+b_0(q-q_0))g(q)dq+\ldots+\int_{q_n}^{q_{n+1}}(y_n+b_n(q-q_n))g(q)dq \\
&= \int_{q_0}^{q_1} y_0\ g(q)dq+\ldots+\int_{q_n}^{q_{n+1}} y_n\ g(q)dq \\
&\quad+\int_{q_0}^{q_1} b_0(q-q_0)g(q)dq+\ldots+\int_{q_n}^{q_{n+1}} b_n(q-q_n)g(q)dq \\
&= y_0K_0+\ldots y_nK_n+b_0(J_0-q_0K_0)+\ldots+b_n(J_n-q_nK_n)
\end{aligned}$$

where, $K_i = \int_{q_i}^{q_{i+1}} g(q)dq$ and $J_i = \int_{q_i}^{q_{i+1}} q\,g(q)dq$, for $i = 0,\ldots,n$. These can be considered as partitions of probability and expectation respectively, in that $\sum_{i=0}^n K_i = 1$ and $\sum_{i=0}^n J_i = Y$.

By defining $A_i = y_iK_i$ and $B_i = b_i(J_i - q_iK_i)$, I can say

$$\mathbb{E}\left[\frac{v(x)}{s}\right] = \sum_{i=0}^n A_i + \sum_{i=0}^n B_i$$

And, finally, letting $A = \sum_{i=0}^n A_i$ and $B = \sum_{i=0}^n B_i$ I have that

$$\mathbb{E}\left[\frac{v(x)}{s}\right] = A + B$$

Therefore

$$\begin{aligned}
\mathbb{E}[r_O] &= \mathbb{E}\left[\frac{v(x)}{s}\right] - (1+r_F)\frac{p}{s} + r_F \\
&= A + B - (1+r_F)\frac{p}{s} + r_F
\end{aligned}$$

Similarly, the second moment is written as

$$\begin{aligned}
\mathbb{E}\left[\left(\frac{v(x)}{s}\right)^2\right] &= \int_{q_0}^{q_1}(y_0+b_0(q-q_0))^2g(q)dq+\ldots+\int_{q_n}^{q_{n+1}}(y_n+b_n(q-q_n)))^2g(q)dq \\
&= \int_{q_0}^{q_1} y_0^2\ g(q)dq+\ldots+\int_{q_n}^{q_{n+1}} y_n^2\ g(q)dq \\
&\quad+\int_{q_0}^{q_1} b_0^2(q-q_0)^2g(q)dq+\ldots+\int_{q_n}^{q_{n+1}} b_n^2(q-q_n)^2g(q)dq \\
&\quad+2\left(\int_{q_0}^{q_1} y_0b_0(q-q_n)g(q)dq+\ldots+\int_{q_n}^{q_{n+1}} y_0b_n(q-q_n)g(q)dq\right) \\
&= \sum_{i=0}^n C_i + \sum_{i=0}^n D_i + 2\sum_{i=0}^n E_i
\end{aligned}$$

where $C_i = \int_{q_i}^{q_{i+1}} y_i^2 g(q)dq$, $D_i = \int_{q_i}^{q_{i+1}} b_i^2(q-q_i)^2g(q)dq$ and $E_i = \int_{q_i}^{q_{i+1}} y_ib_i(q-q_i)g(q)dq$.

I can immediately write that $C_i = y_i^2K_i$ and $E_i = y_ib_i(J_i - q_iK_i)$ for $i = 0,\ldots,n$.

For $D_0,\ldots,D_n$ I expand the squared term of the integrand giving me that $D_i = \int_{q_i}^{q_{i+1}} b_i^2(q^2-2qq_i+q_i^2)g(q)dq$. Defining $L_i = \int_{q_i}^{q_{i+1}} q^2g(q)dq$ (which can be considered as a partition of the second moment satisfying $\sum_{i=0}^{n} L_i = Z$), I can write that, for $i = 0, \ldots, n$, $D_i = b_i^2(L_i - 2q_i J_i + q_i^2 K_i)$.

Finally, letting $C = \sum_{i=0}^{n} C_i$, $D = \sum_{i=0}^{n} D_i$ and $E = \sum_{i=0}^{n} E_i$, I have that

$$\mathbb{E}\left[\left(\frac{v(x)}{s}\right)^2\right] = C + D + 2E$$

Combining these two expressions gives us that

$$\mathbb{V}[r_O] = \mathbb{V}\left[\frac{v(x)}{s}\right] = \mathbb{E}\left[\left(\frac{v(x)}{s}\right)^2\right] - \mathbb{E}\left[\frac{v(x)}{s}\right]^2 = C + D + 2E - (A+B)^2$$

Also,

$$\begin{aligned}
\mathbb{E}\left[\frac{v(x)}{s}.\frac{x}{s}\right] &= \int_{q_0}^{q_1} (y_0 + b_0(q - q_0))q^2 g(q)dq + \ldots + \int_{q_n}^{q_{n+1}} (y_n + b_n(q - q_n))q^2 g(q)dq \\
&= \int_{q_0}^{q_1} y_0\ q\ g(q)dq + \ldots + \int_{q_n}^{q_{n+1}} y_n\ q\ g(q)dq \\
&\quad + \int_{q_0}^{q_1} b_0(q - q_0)q\ g(q)dq + \ldots + \int_{q_n}^{q_{n+1}} b_n(q - q_n)q\ g(q)dq \\
&= \sum_{i=0}^{n} F_i + \sum_{i=0}^{n} G_i
\end{aligned}$$

where $F_i = \int_{q_i}^{q_{i+1}} y_i q\ g(q)dq$ and $G_i = \int_{q_i}^{q_{i+1}} b_i(q - q_i)q\ g(q)dq$. Using the above definitions, I write $F_i = y_i J_i$ and $G_i = b_i(L_i - q_i J_i)$. Letting $F = \sum_{i=0}^{n} F_i$ and $G = \sum_{i=0}^{n} G_i$

$$\mathbb{E}\left[\frac{v(x)}{s}.\frac{x}{s}\right] = F + G$$

Therefore

$$\mathbb{COV}[r_O, r_{/!A}] = \mathbb{E}\left[\frac{v(x)}{s}.\frac{x}{s}\right] - \mathbb{E}\left[\frac{v(x)}{s}\right]\mathbb{E}\left[\frac{x}{s}\right] = F + G - Y(A+B)$$

as required. □

Property A.21 (Statistics under geometric Brownian motion). When the underlying asset price follows a geometric Brownian motion process with instantaneous mean μ, and instantaneous standard deviation σ, then

$$Y = \exp\{\mu t\} \qquad \text{and} \qquad Z = \exp\{(2\mu + \sigma^2)t\}$$

where t denotes the length of time of the investment period, and

$$\begin{aligned}
K_i &= \Phi(u_{i+1}) - \Phi(u_i) \\
J_i &= Y(\Phi(v_{i+1}) - \Phi(v_i)) \\
L_i &= Z(\Phi(w_{i+1}) - \Phi(w_i))
\end{aligned}$$

where

$$\begin{aligned} u_i &= \frac{\log(q_i) - (\mu - {}^1\!/\!_2\, \sigma^2)t}{\sigma\sqrt{t}} \\ v_i &= \frac{\log(q_i) - (\mu + {}^1\!/\!_2\, \sigma^2)t}{\sigma\sqrt{t}} \\ w_i &= \frac{\log(q_i) - (\mu + {}^3\!/\!_2\, \sigma^2)t}{\sigma\sqrt{t}} \end{aligned}$$

Proof. If the asset price follows geometric Brownian motion with instantaneous mean μ, and instantaneous standard deviation σ, then q, is log-normally distributed. Specifically $\log(q)$ is normally distributed with mean $\theta = (\mu - {}^1\!/\!_2\, \sigma^2)t$ and variance $\omega^2 = \sigma^2 t$.

Letting $y = (\log(q) - \theta)/\omega$ we can write

$$K_i = \int_{q_i}^{q_{i+1}} g(q)dq = \int_{u_i}^{u_{i+1}} \frac{1}{\sqrt{2\pi}} \exp\{-{}^1\!/\!_2\, y^2\}dy$$

where $u_i = (\log(q_i) - \theta)/\omega$, which can be expressed as

$$u_i = \frac{\log(qi) - \theta}{\omega} = \frac{\log(q_i) - (\mu - {}^1\!/\!_2\, \sigma^2)t}{\sigma\sqrt{t}}$$

Therefore, $K_i = \Phi(u_{i+1}) - \Phi(u_i)$
where $\Phi(z)$ is the cumulative distribution function of a standard normal distribution. That is

$$\Phi(z) = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} \exp\{-{}^1\!/\!_2 y^2\}dy$$

Similarly,

$$\begin{aligned} J_i = \int_{q_i}^{q_{i+1}} q\; g(q)dq &= \int_{u_i}^{u_{i+1}} \exp\{y\omega + \theta\} \frac{1}{\sqrt{2\pi}} \exp\{-{}^1\!/\!_2\, y^2\}dy \\ &= \exp\{{}^1\!/\!_2\, \omega^2 + \theta\} \int_{u_i}^{u_{i+1}} \frac{1}{\sqrt{2\pi}} \exp\{-{}^1\!/\!_2\, (y-\omega)^2\}dy \end{aligned}$$

Letting $z = y - \omega$, we can write

$$J_i = \exp\{{}^1\!/\!_2\, \omega^2 + \theta\} \int_{v_i}^{v_{i+1}} \frac{1}{\sqrt{2\pi}} \exp\{-{}^1\!/\!_2\, z^2\}dz$$

where $v_i = u_i - \omega$. which can be expressed as

$$v_i = \frac{\log(q_i) - (\mu + {}^1\!/\!_2\, \sigma^2)t}{\sigma\sqrt{t}}$$

Noting that $\exp\{{}^1\!/\!_2\, \omega^2 + \theta\} = \exp\{\mu t\}$, and using identical logic to the above, we also have that

$$Y = \int_0^{\infty} q\; g(q)dq = \exp\{{}^1\!/\!_2\, \omega^2 + \theta\} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\{-{}^1\!/\!_2\, z^2\}dz = \exp\{\mu t\}$$

Therefore, $J_i = Y(\Phi(v_{i+1}) - \Phi(v_i))$.

Similarly,

$$
\begin{aligned}
L_i = \int_{q_i}^{q_{i+1}} q^2\, g(q) dq &= \int_{u_i}^{u_{i+1}} \exp\{y\omega + \theta\}^2 \frac{1}{\sqrt{2\pi}} \exp\{-{}^1\!/\!{}_2\, y^2\} dy \\
&= \exp\{2\omega^2 + 2\theta\} \int_{u_i}^{u_{i+1}} \frac{1}{\sqrt{2\pi}} \exp\{-{}^1\!/\!{}_2\, (y - 2\omega)^2\} dy
\end{aligned}
$$

Letting $z = y - 2\omega$, we can write

$$L_i = \exp\{2\omega^2 + 2\theta\} \int_{w_i}^{w_{i+1}} \frac{1}{\sqrt{2\pi}} \exp\{-{}^1\!/\!{}_2\, z^2\} dz$$

where $w_i = u_i - 2\omega$, which can be expressed as

$$w_i = u_i - 2\omega = \frac{\log(q_i) - (\mu + {}^3\!/\!{}_2\, \sigma^2)t}{\sigma\sqrt{t}}$$

Noting that $\exp\{2\omega^2 + 2\theta\} = \exp\{(2\mu + \sigma^2)t\}$, and using identical logic to the above, we also have that

$$Z = \int_0^{\infty} q^2\, g(q) dq = \exp\{2\omega^2 + 2\theta\} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\{-{}^1\!/\!{}_2\, z^2\} dz = \exp\{(2\mu + \sigma^2)t\}$$

Therefore, $L_i = Z(\Phi(w_{i+1}) - \Phi(w_i))$. □

Definition A.22. The *instantaneous risk-free rate*, denoted by r_f, satisfies $r_f = \frac{1}{t}\log(1 + r_F)$.

Property A.23 (Option price under geometric Brownian motion). When the underlying asset price follows a geometric Brownian motion process with instantaneous mean μ, and instantaneous standard deviation σ, then $(1 + r_F)\frac{p}{s} = A^* + B^*$ where A^* and B^* are computed exactly as A and B were described in the *Mean, variance and covariance of an option adjusted portfolio return* property, but with μ replaced throughout by the instantaneous risk-free rate r_f and σ by the implied volatility of the call option with the ith strike price.

Proof. If the asset price follows a geometric Brownian motion process, then a risk-neutral portfolio can be formed[1]. Therefore, the options can be priced using risk-neutral valuation with the market's price of volatility. Mathematically, this is expressed as

$$(1 + r_F)\frac{p}{s} = \mathbb{E}^*\left[\frac{v(x)}{s}\right]$$

where the $\mathbb{E}^*[.]$ represents the expectation operator using the risk-neutral distribution. Under the risk-neutral distribution $\log(q)$ is normally distributed with mean $(r_f - {}^1\!/\!{}_2\, \sigma^{*2})t$ and variance $\sigma^{*2}t$, where σ^* denotes the implied volatility. Following the logic of the *Mean, variance and covariance of an option adjusted portfolio return* property, I can write that

$$\mathbb{E}^*\left[\frac{v(x)}{s}\right] = \sum_{i=0}^{n} A_i^* + \sum_{i=0}^{n} B_i^*$$

where $A_i^* = y_i K_i^*$ and $B_i^* = b_i(J_i^* - q_i K_i^*)$, for $i = 0, \ldots, n$, and

$$K_i^* = \int_{q_i}^{q_{i+1}} g(q)dq \qquad \text{and} \qquad J_i^* = \int_{q_i}^{q_{i+1}} q\, g(q)dq.$$

Furthermore, I can use the *Statistics under geometric Brownian motion* property to say

$$\begin{aligned} K_i^* &= \Phi(u_{i+1}^*) - \Phi(u_i^*) \\ J_i^* &= Y^*(\Phi(v_{i+1}^*) - \Phi(v_i^*)) \end{aligned}$$

where

$$\begin{aligned} u_i^* &= \frac{\log(q_i) - (r_f - 1/2\, \sigma^{*2})t}{\sigma^* \sqrt{t}} \\ v_i^* &= \frac{\log(q_i) - (r_f + 1/2\, \sigma^{*2})t}{\sigma^* \sqrt{t}} \end{aligned}$$

with $Y^* = \exp\{r_f t\}$. □

Property A.24 (Covariance of one option adjusted asset with another asset). The return of an option adjusted asset (with a value function, $v^{(1)}$, defined by intercepts $y_1^{(1)}, \ldots, y_n^{(1)}$, slopes $b_1^{(1)}, \ldots, b_n^{(1)}$, and strikes $q_1^{(1)}, \ldots, q_n^{(1)}$ on an underlying asset with start and end prices of $s^{(1)}$ and $x^{(1)}$ respectively) have a covariance with the return of another, non-option adjusted, asset (with start and end prices of $s^{(2)}$ and $x^{(2)}$ respectively) given by

$$\mathbb{COV}[r_O^{(1)}, r_A^{(2)}] = \mathbb{E}\left[\frac{v^{(1)}(x^{(1)})}{s^{(1)}} . \frac{x^{(2)}}{s^{(2)}}\right] - \mathbb{E}\left[\frac{v^{(1)}(x^{(1)})}{s^{(1)}}\right] \mathbb{E}\left[\frac{x^{(2)}}{s^{(2)}}\right]$$

where

$$\mathbb{E}\left[\frac{v^{(1)}(x^{(1)})}{s^{(1)}} . \frac{x^{(2)}}{s^{(2)}}\right] = D + E$$

where $D = \sum_{i=0}^n D_i$ and $E = \sum_{i=0}^n E_i$
with $D_i = b_i^{(1)}(L_i - q_i^{(1)} J_i)$ and $E_i = y_i^{(1)} J_i$ for $i = 0, \ldots, n$.

$$\begin{aligned} J_i &= Y^{(2)}(\Phi(\psi_{i+1}) - \Phi(\psi_i)) \\ L_i &= Y^{(12)}(\Phi(\xi_{i+1}) - \Phi(\xi_i)) \end{aligned}$$

where

$$\begin{aligned} \psi_i &= \frac{\log(q_i^{(1)}) - (\mu^{(1)} - 1/2\, (\sigma^{(1)2} - 2\rho\sigma^{(1)}\sigma^{(2)}))t}{\sigma^{(1)}\sqrt{t}} \\ \xi_i &= \frac{\log(q_i^{(1)}) - (\mu^{(1)} + 1/2\, (\sigma^{(1)2} + 2\rho\sigma^{(1)}\sigma^{(2)}))t}{\sigma^{(1)}\sqrt{t}} \end{aligned}$$

and where $Y^{(1)} = \exp\{\mu^{(1)} t\}$, $Y^{(2)} = \exp\{\mu^{(2)} t\}$, and $Y^{(12)} = Y^{(1)} Y^{(2)} \exp\{\rho\sigma^{(1)}\sigma^{(2)} t\}$.

The variables $\mu^{(1)}$ and $\sigma^{(1)}$ denote the instantaneous mean and standard deviation of the price process of the underlying asset in the option adjusted asset, and

$\mu^{(2)}$ and $\sigma^{(2)}$ denote the instantaneous mean and standard deviation of the price process of the other asset, and ρ denotes their instantaneous correlation.

Proof. The option adjusted asset has value function denoted by $v^{(1)}$. Its return is

$$r_O^{(1)} = \frac{v^{(1)}(x^{(1)})}{s^{(1)}} - (1 + r_F)\frac{p^{(1)}}{s^{(1)}} + r_F$$

Therefore

$$r_O^{(1)} - \mathbb{E}[r_O^{(1)}] = \frac{v^{(1)}(x^{(1)})}{s^{(1)}} - \mathbb{E}\left[\frac{v^{(1)}(x^{(1)})}{s^{(1)}}\right]$$

The covariance of this return with another asset is therefore

$$\begin{aligned}
\mathbb{COV}[r_O^{(1)}, r_A^{(2)}] &= \mathbb{E}\left[\left(r_O^{(1)} - \mathbb{E}[r_O^{(1)}]\right)\left(r_A^{(2)} - \mathbb{E}[r_A^{(2)}]\right)\right] \\
&= \mathbb{E}\left[\left(\frac{v^{(1)}(x^{(1)})}{s^{(1)}} - \mathbb{E}\left[\frac{v^{(1)}(x^{(1)})}{s^{(1)}}\right]\right)\left(\frac{x^{(2)}}{s^{(2)}} - \mathbb{E}\left[\frac{x^{(2)}}{s^{(2)}}\right]\right)\right] \\
&= \mathbb{E}\left[\frac{v^{(1)}(x^{(1)})}{s^{(1)}} . \frac{x^{(2)}}{s^{(2)}}\right] - \mathbb{E}\left[\frac{v^{(1)}(x^{(1)})}{s^{(1)}}\right]\mathbb{E}\left[\frac{x^{(2)}}{s^{(2)}}\right]
\end{aligned}$$

Now,

$$\begin{aligned}
\mathbb{E}\left[\frac{v^{(1)}(x^{(1)})}{s^{(1)}} . \frac{x^{(2)}}{s^{(2)}}\right] &= \int_{q_0^{(1)}}^{q_1^{(1)}} \int_0^\infty (y_0^{(1)} + b_0^{(1)}(q^{(1)} - q_0^{(1)}))q^{(2)} g(q^{(1)}, q^{(2)}) dq^{(2)} dq^{(1)} \\
&\qquad + \\
&\qquad \vdots \\
&\qquad + \\
&\quad \int_{q_n^{(1)}}^{q_{n+1}^{(1)}} \int_0^\infty (y_n^{(1)} + b_n^{(1)}(q^{(1)} - q_n^{(1)}))q^{(2)} g(q^{(1)}, q^{(2)}) dq^{(2)} dq^{(1)} \\
&= \sum_{i=0}^{n} y_i^{(1)} J_i + \sum_{i=0}^{n} b_i^{(1)}(L_i - q_i^{(1)} J_i)
\end{aligned}$$

where

$$\begin{aligned}
J_i &= \int_{q_i^{(1)}}^{q_{i+1}^{(1)}} \int_0^\infty q^{(2)}\ g(q^{(1)}, q^{(2)}) dq^{(2)} dq^{(1)} \\
L_i &= \int_{q_i^{(1)}}^{q_{i+1}^{(1)}} \int_0^\infty q^{(1)}\ q^{(2)}\ g(q^{(1)}, q^{(2)}) dq^{(2)} dq^{(1)}
\end{aligned}$$

I can express this as

$$\mathbb{E}\left[\frac{v^{(1)}(x^{(1)})}{s^{(1)}} . \frac{x^{(2)}}{s^{(2)}}\right] = D + E$$

where $D = \sum_{i=0}^{n} D_i$ and $E = \sum_{i=0}^{n} E_i$
with $D_i = b_i^{(1)}(L_i - q_i^{(1)} J_i)$ and $E_i = y_i^{(1)} J_i$.

If $q^{(1)}$ and $q^{(2)}$ follow the bivariate the geometric Brownian motion process defined by

$$\begin{aligned} dq^{(1)} &= q^{(1)}\mu^{(1)}dt + q^{(1)}\sigma^{(1)}dW^{(1)} \\ dq^{(2)} &= q^{(2)}\mu^{(2)}dt + q^{(2)}\sigma^{(2)}(\rho dW^{(1)} + \gamma dW^{(2)}) \end{aligned}$$

where $\gamma = \sqrt{1-\rho^2}$, then, for $k = 1$ and 2, $\log(q^{(k)})$ is distributed normally with mean $\theta^{(k)}$ and variance $\omega^{(k)2}$, with correlation ρ, where

$$\theta^{(k)} = (\mu^{(k)} - {}^1\!/\!_2\, \sigma^{(k)2})t \qquad \text{and} \qquad \omega^{(k)} = \sigma^{(k)}\sqrt{t}$$

In this case we have that

$$\begin{aligned} J_i &= \int_{q_i^{(1)}}^{q_{i+1}^{(1)}} \int_0^{\infty} q^{(2)} g(q^{(1)}, q^{(2)}) dq^{(2)} dq^{(1)} \\ &= \int_{u_i^{(1)}}^{u_{i+1}^{(1)}} \int_0^{\infty} \exp\{\omega^{(2)} y^{(2)} + \theta^{(2)}\} \\ &\quad \times \frac{1}{2\pi\gamma} \exp\{-\frac{1}{2\gamma^2}(y^{(1)2} - 2\rho y^{(1)} y^{(2)} + y^{(2)2})\} dy^{(2)} dy^{(1)} \\ &= Y^{(2)} \int_{u_i^{(1)}}^{u_{i+1}^{(1)}} \frac{1}{\sqrt{2\pi}} \exp\{-{}^1\!/\!_2\, (y^{(1)} - \rho\omega^{(2)})^2\} dy^{(1)} \end{aligned}$$

where

$$u_i^{(1)} = \frac{\log(q_i^{(1)}) - (\mu^{(1)} - {}^1\!/\!_2\, \sigma^{(1)2})t}{\sigma^{(1)}\sqrt{t}}$$

and where $Y^{(2)} = \exp\{\mu^{(2)} t\}$.

Let $z = y^{(1)} - \rho\omega^{(2)}$, then

$$J_i = Y^{(2)} \int_{\psi_i^{(1)}}^{\nu_{i+1}^{(1)}} \frac{1}{\sqrt{2\pi}} \exp\{-{}^1\!/\!_2\, z^2\} dz$$

where $\psi_i^{(1)} = u_i^{(1)} - \rho\omega^{(2)}$, which can be expressed as

$$\psi_i^{(1)} = u_i^{(1)} - \omega^{(1)} = \frac{\log(q_i^{(1)}) - (\mu^{(1)} - {}^1\!/\!_2\, (\sigma^{(1)2} - 2\rho\sigma^{(1)}\sigma^{(2)}))t}{\sigma^{(1)}\sqrt{t}}$$

Therefore, $J_i = Y^{(2)}(\Phi(\psi_{i+1}^{(1)}) - \Phi(\psi_i^{(1)}))$.

We also have that

$$
\begin{aligned}
L_i &= \int_{q_i^{(1)}}^{q_{i+1}^{(1)}} \int_0^\infty q^{(1)} q^{(2)} g(q^{(1)}, q^{(2)}) dq^{(2)} dq^{(1)} \\
&= \int_{u_i^{(1)}}^{u_{i+1}^{(1)}} \int_0^\infty \exp\{\omega^{(1)} y^{(1)} + \theta^{(1)}\} exp\{\omega^{(2)} y^{(2)} + \theta^{(2)}\} \\
&\quad \times \frac{1}{2\pi\gamma} \exp\{-\frac{1}{2\gamma^2}(y^{(1)2} - 2\rho y^{(1)} y^{(2)} + y^{(2)2})\} dy^{(2)} dy^{(1)} \\
&= Y^{(12)} \int_{u_i^{(1)}}^{u_{i+1}^{(1)}} \frac{1}{\sqrt{2\pi}} \exp\{-{}^1\!/_2\ (y^{(1)} - (\rho\omega^{(2)} + \omega^{(1)}))^2\} dy^{(1)}
\end{aligned}
$$

where $Y^{(12}) = Y^{(1)} Y^{(2)} \exp\{\rho\sigma^{(1)}\sigma^{(2)} t\}$.

Let $z = y^{(1)} - \omega^{(1)} - \rho\omega^{(2)}$, then

$$
L_i = Y^{(12)} \int_{\xi_i^{(1)}}^{\xi_{i+1}^{(1)}} \frac{1}{\sqrt{2\pi}} \exp\{-{}^1\!/_2\ z^2)\} dz
$$

where $\xi_i^{(1)} = u_i^{(1)} - \omega^{(1)} - \rho\omega^{(2)}$, which can be expressed as

$$
\xi_i^{(1)} = u_i^{(1)} - \omega^{(1)} - \rho\omega^{(2)} = \frac{\log(q_i^{(1)}) - (\mu^{(1)} + {}^1\!/_2\ (\sigma^{(1)2} + 2\rho\sigma^{(1)}\sigma^{(2)}))t}{\sigma^{(1)}\sqrt{t}}
$$

Therefore, $L_i = Y^{(12)}(\Phi(\xi_{i+1}^{(1)}) - \Phi(\xi_i^{(1)}))$ □

Property A.25 (Covariance of two option adjusted assets). Consider two option adjusted assets. The first has a value function, $v^{(1)}$, defined by intercepts $y_1^{(1)}, \ldots, y_{n^{(1)}}^{(1)}$, slopes $b_1^{(1)}, \ldots, b_{n^{(1)}}^{(1)}$, and strikes $q_1^{(1)}, \ldots, q_{n^{(1)}}^{(1)}$, on an underlying asset with start and end prices of $s^{(1)}$ and $x^{(1)}$ respectively. The second has a value function defined by has a value function, $v^{(2)}$, defined by intercepts $y_1^{(2)}, \ldots, y_{n^{(2)}}^{(2)}$, slopes $b_1^{(2)}, \ldots, b_{n^{(2)}}^{(2)}$, and strikes $q_1^{(1)}, \ldots, q_{n^{(2)}}^{(2)}$ on an underlying asset with start and end prices of $s^{(2)}$ and $x^{(2)}$ respectively.

The covariance of returns of two option adjusted assets is

$$
\mathbb{COV}[r_O^{(1)}, r_O^{(2)}] = \mathbb{E}\left[\frac{v^{(1)}(x^{(1)})}{s^{(1)}}.\frac{v^{(2)}(x^{(2)})}{s^{(2)}}\right] - \mathbb{E}\left[\frac{v^{(1)}(x^{(1)})}{s^{(1)}}\right] \mathbb{E}\left[\frac{v^{(2)}(x^{(2)})}{s^{(2)}}\right]
$$

where

$$
\mathbb{E}\left[\frac{v^{(1)}(x^{(1)})}{s^{(1)}}.\frac{v^{(2)}(x^{(2)})}{s^{(2)}}\right] = C + D + E1 + E2
$$

where $C = \sum_{i=0}^{n^{(1)}} \sum_{j=0}^{n^{(2)}} C_{ij}$, $D = \sum_{i=0}^{n^{(1)}} \sum_{j=0}^{n^{(2)}} D_{ij}$, $E1 = \sum_{i=0}^{n^{(1)}} \sum_{j=0}^{n^{(2)}} E1_{ij}$, and $E2 = \sum_{i=0}^{n^{(1)}} \sum_{j=0}^{n^{(2)}} E2_{ij}$
with

$$
\begin{aligned}
C_{ij} &= y_i^{(1)} y_j^{(2)} K_{ij} \\
D_{ij} &= b_i^{(1)} b_j^{(2)} (L_{ii} - q_j^{(2)} J_{ij}^{(1)} - q_i^{(1)} J_{ij}^{(2)} + q_i^{(1)} q_j^{(2)} K_{ij}) \\
E1_{ij} &= y_i^{(1)} b_j^{(2)} (J_{ij}^{(2)} - q_j^{(2)} K_{ij}) \\
E2_{ij} &= y_j^{(2)} b_i^{(1)} (J_{ij}^{(1)} - q_i^{(1)} K_{ij})
\end{aligned}
$$

and

$$
\begin{aligned}
K_{ij} &= \Phi_\rho(u_{i+1}^{(1)}, u_{j+1}^{(2)}) - \Phi_\rho(u_{i+1}^{(1)}, u_j^{(2)}) - \Phi_\rho(u_i^{(1)}, u_{j+1}^{(2)}) + \Phi_\rho(u_i^{(1)}, u_j^{(2)}) \\
J_{ij}^{(1)} &= Y^{(1)}(\Phi_\rho(v_{i+1}^{(1)}, \psi_{j+1}^{(2)}) - \Phi_\rho(v_{i+1}^{(1)}, \psi_j^{(2)}) - \Phi_\rho(v_i^{(1)}, \psi_{j+1}^{(2)}) + \Phi_\rho(v_i^{(1)}, \psi_j^{(2)})) \\
J_{ij}^{(2)} &= Y^{(2)}(\Phi_\rho(\psi_{i+1}^{(1)}, v_{j+1}^{(2)}) - \Phi_\rho(\psi_{i+1}^{(1)}, v_j^{(2)}) - \Phi_\rho(\psi_i^{(1)}, v_{j+1}^{(2)}) + \Phi_\rho(\psi_i^{(1)}, v_j^{(2)})) \\
L_{ij} &= Y^{(12)}(\Phi_\rho(\xi_{i+1}^{(1)}, \xi_{j+1}^{(2)}) - \Phi_\rho(\xi_{i+1}^{(1)}, \xi_j^{(2)}) - \Phi_\rho(\xi_i^{(1)}, \xi_{j+1}^{(2)}) + \Phi_\rho(\xi_i^{(1)}, \xi_j^{(2)}))
\end{aligned}
$$

where, for $k = 1$ and 2,

$$
\begin{aligned}
\psi_i^{(k)} &= \frac{\log(q_i^{(k)}) - (\mu^{(k)} - 1/2\,(\sigma^{(k)2} - 2\rho\sigma^{(1)}\sigma^{(2)}))t}{\sigma^{(k)}\sqrt{t}} \\
\xi_i^{(k)} &= \frac{\log(q_i^{(k)}) - (\mu^{(k)} + 1/2\,(\sigma^{(k)2} + 2\rho\sigma^{(1)}\sigma^{(2)}))t}{\sigma^{(k)}\sqrt{t}}
\end{aligned}
$$

and $Y^{(1)} = \exp\{\mu^{(1)}t\}$, $Y^{(2)} = \exp\{\mu^{(2)}t\}$ and $Y_{(12)} = Y^{(1)}Y^{(2)}\exp\{\rho\sigma^{(1)}\sigma^{(2)}t\}$. The function $\Phi_\rho(x, y)$ is the cumulative distribution function of the bivariate standard normal distribution with correlation ρ.

The variables $\mu^{(1)}$ and $\sigma^{(1)}$ denote the instantaneous mean and standard deviation of the price process of the underlying asset in the option adjusted asset, and $\mu^{(2)}$ and $\sigma^{(2)}$ denote the instantaneous mean and standard deviation of the price process of the other asset, and ρ denotes their instantaneous correlation.

Proof. For asset k ($k = 1$ and 2), the option adjusted return is

$$
r_O^{(k)} = \frac{v^{(k)}(x^{(k)})}{s^{(k)}} - (1 + r_F)\frac{p^{(k)}}{s^{(k)}} + r_F
$$

Therefore

$$
r_O^{(k)} - \mathbb{E}[r_O^{(k)}] = \frac{v^{(k)}(x^{(k)})}{s^{(k)}} - \mathbb{E}\left[\frac{v^{(k)}(x^{(k)})}{s^{(k)}}\right]
$$

The covariance of these returns is therefore

$$
\begin{aligned}
\mathbb{COV}[r_O^{(1)}, r_O^{(2)}] &= \mathbb{E}\left[(r_O^{(1)} - \mathbb{E}[r_O^{(1)}])(r_O^{(2)} - \mathbb{E}[r_O^{(2)}])\right] \\
&= \mathbb{E}\left[\left(\frac{v^{(1)}(x^{(1)})}{s^{(1)}} - \mathbb{E}\left[\frac{v^{(1)}(x^{(1)})}{s^{(1)}}\right]\right)\left(\frac{v^{(2)}(x^{(2)})}{s^{(2)}} - \mathbb{E}\left[\frac{v^{(2)}(x^{(2)})}{s^{(2)}}\right]\right)\right] \\
&= \mathbb{E}\left[\frac{v^{(1)}(x^{(1)})}{s^{(1)}}.\frac{v^{(2)}(x^{(2)})}{s^{(2)}}\right] - \mathbb{E}\left[\frac{v^{(1)}(x^{(1)})}{s^{(1)}}\right]\mathbb{E}\left[\frac{v^{(2)}(x^{(2)})}{s^{(2)}}\right]
\end{aligned}
$$

Now,

$$\mathbb{E}\left[\frac{v^{(1)}(x^{(1)})}{s^{(1)}}.\frac{v^{(2)}(x^{(2)})}{s^{(2)}}\right] =$$

$$\int_{q_0^{(1)}}^{q_1^{(1)}}\int_{q_0^{(2)}}^{q_1^{(2)}}\left(y_0^{(1)}+b_0^{(1)}(q^{(1)}-q_0^{(1)})\right)\left(y_0^{(2)}+b_0^{(2)}(q^{(2)}-q_0^{(2)})\right)g(q^{(1)},q^{(2)})dq^{(2)}dq^{(1)}$$

$$+$$

$$\vdots$$

$$+$$

$$\int_{q_{n^{(1)}}^{(1)}}^{\infty}\int_{q_{n^{(2)}}^{(2)}}^{\infty}\left(y_{n^{(1)}}^{(1)}+b_{n^{(1)}}^{(1)}(q^{(1)}-q_{n^{(1)}}^{(1)})\right)\left(y_{n^{(2)}}^{(2)}+b_{n^{(2)}}^{(2)}(q^{(2)}-q_{n^{(2)}}^{(2)})\right)g(q^{(1)},q^{(2)})dq^{(2)}dq^{(1)}$$

$$\begin{aligned}
= \quad & \sum_{i=0}^{n^{(1)}}\sum_{j=0}^{n^{(2)}} y_i^{(1)}y_j^{(2)}K_{ij} \\
& + \sum_{i=0}^{n^{(1)}}\sum_{j=0}^{n^{(2)}} y_i^{(1)}b_j^{(2)}(J_{ij}^{(2)}-q_j^{(2)}K_{ij}) \\
& + \sum_{i=0}^{n^{(1)}}\sum_{j=0}^{n^{(2)}} y_j^{(2)}b_i^{(1)}(J_{ij}^{(1)}-q_i^{(1)}K_{ij}) \\
& + \sum_{i=0}^{n^{(1)}}\sum_{j=0}^{n^{(2)}} b_i^{(1)}b_j^{(2)}(L_{ij}-q_j^{(2)}J_{ij}^{(1)}-q_i^{(1)}J_{ij}^{(2)}+q_i^{(1)}q_j^{(2)}K_{ij})
\end{aligned}$$

where

$$\begin{aligned}
K_{ij} &= \int_{q_i^{(1)}}^{q_{i+1}^{(1)}}\int_{q_j^{(2)}}^{q_{j+1}^{(2)}} g(q^{(1)},q^{(2)})dq^{(2)}dq^{(1)} \\
J_{ij}^{(1)} &= \int_{q_i^{(1)}}^{q_{i+1}^{(1)}}\int_{q_j^{(2)}}^{q_{j+1}^{(2)}} q^{(1)}g(q^{(1)},q^{(2)})dq^{(2)}dq^{(1)} \\
J_{ij}^{(2)} &= \int_{q_i^{(1)}}^{q_{i+1}^{(1)}}\int_{q_j^{(2)}}^{q_{j+1}^{(2)}} q^{(2)}g(q^{(1)},q^{(2)})dq^{(2)}dq^{(1)} \\
L_{ij} &= \int_{q_i^{(1)}}^{q_{i+1}^{(1)}}\int_{q_j^{(2)}}^{q_{j+1}^{(2)}} q^{(1)}q^{(2)}g(q^{(1)},q^{(2)})dq^{(2)}dq^{(1)}
\end{aligned}$$

I can express this as

$$\mathbb{E}\left[\frac{v^{(1)}(x^{(1)})}{s^{(1)}}.\frac{v^{(2)}(x^{(2)})}{s^{(2)}}\right] = C+D+E1+E2$$

where $C = \sum_{i=0}^{n^{(1)}}\sum_{j=0}^{n^{(2)}} C_{ij}$, $D = \sum_{i=0}^{n^{(1)}}\sum_{j=0}^{n^{(2)}} D_{ij}$, $E1 = \sum_{i=0}^{n^{(1)}}\sum_{j=0}^{n^{(2)}} E1_{ij}$, and $E2 = \sum_{i=0}^{n^{(1)}}\sum_{j=0}^{n^{(2)}} E2_{ij}$
with

$$\begin{aligned}
C_{ij} &= y_i^{(1)} y_j^{(2)} K_{ij} \\
D_{ij} &= b_i^{(1)} b_j^{(2)} (L_{ii} - q_j^{(2)} J_{ij}^{(1)} - q_i^{(1)} J_{ij}^{(2)} + q_i^{(1)} q_j^{(2)} K_{ij}) \\
E1_{ij} &= y_i^{(1)} b_j^{(2)} (J_{ij}^{(2)} - q_j^{(2)} K_{ij}) \\
E2_{ij} &= y_j^{(2)} b_i^{(1)} (J_{ij}^{(1)} - q_i^{(1)} K_{ij})
\end{aligned}$$

If $q^{(1)}$ and $q^{(2)}$ follow the bivariate the geometric Brownian motion process defined by

$$\begin{aligned}
dq^{(1)} &= q^{(1)} \mu^{(1)} dt + q^{(1)} \sigma^{(1)} dW^{(1)} \\
dq^{(2)} &= q^{(2)} \mu^{(2)} dt + q^{(2)} \sigma^{(2)} (\rho dW^{(1)} + \gamma dW^{(2)})
\end{aligned}$$

where $\gamma = \sqrt{1-\rho^2}$, then, for $k = 1$ and 2, $\log(q^{(k)})$ is distributed normally with mean $\theta^{(k)}$ and variance $\omega^{(k)2}$, with correlation ρ, where

$$\theta^{(k)} = (\mu^{(k)} - 1/2\ \sigma^{(k)2})t \qquad \text{and} \qquad \omega^{(k)} = \sigma^{(k)} \sqrt{t}$$

In this case we have that

$$\begin{aligned}
K_{ij} &= \int_{q_i^{(1)}}^{q_{i+1}^{(1)}} \int_{q_j^{(2)}}^{q_{j+1}^{(2)}} g(q^{(1)}, q^{(2)}) dq^{(1)} dq^{(2)} \\
&= \int_{u_i^{(1)}}^{u_{i+1}^{(1)}} \int_{u_j^{(2)}}^{u_{j+1}^{(2)}} \frac{1}{2\pi\gamma} \exp\{-\frac{1}{2\gamma^2}\ (z^{(1)2} - 2\rho z^{(1)} z^{(2)} + z^{(2)2})\} dz^{(1)} dz^{(2)}
\end{aligned}$$

where $u_i^{(k)} = (\log(q_i^{(k)}) - \theta^{(k)})/\omega^{(k)}$, which can be expressed as

$$u_i^{(k)} = \frac{\log(q_i^{(k)}) - (\mu^{(k)} - 1/2\ \sigma^{(k)2})t}{\sigma^{(k)} \sqrt{t}}$$

Therefore

$$K_{ij} = \Phi_\rho(u_{i+1}^{(1)}, u_{j+1}^{(2)}) - \Phi_\rho(u_{i+1}^{(1)}, u_j^{(2)}) - \Phi_\rho(u_i^{(1)}, u_{j+1}^{(2)}) + \Phi_\rho(u_i^{(1)}, u_j^{(2)})$$

where $\Phi_\rho(x, y)$ is the cumulative distribution function of the bivariate standard normal distribution with correlation ρ. That is

$$\Phi_\rho(x, y) = \int_{-\infty}^{x} \int_{-\infty}^{y} \frac{1}{2\pi\gamma} \exp\{-\frac{1}{2\gamma^2}\ (x^2 - 2\rho xy + y^2)\} dx dy$$

$$\begin{aligned}
J_{ij}^{(1)} &= \int_{q_i^{(1)}}^{q_{i+1}^{(1)}} \int_{q_j^{(2)}}^{q_{j+1}^{(2)}} q^{(1)} g(q^{(1)}, q^{(2)}) dq^{(2)} dq^{(1)} \\
&= \int_{u_i^{(1)}}^{u_{i+1}^{(1)}} \int_{u_j^{(2)}}^{u_{j+1}^{(2)}} \exp\{\omega^{(1)} y^{(1)} + \theta^{(1)}\} \\
&\quad \times \frac{1}{2\pi\gamma} \exp\{-\frac{1}{2\gamma^2}\ (y^{(1)2} - 2\rho y^{(1)} y^{(2)} + y^{(2)2})\} dy^{(2)} dy^{(1)}
\end{aligned}$$

Let $z^{(1)} = y^{(1)} - \omega^{(1)}$ and $z^{(2)} = y^{(2)} - \rho\omega^{(2)}$, then

$$
\begin{aligned}
J_{ij}^{(1)} &= \exp\{\theta^{(1)} + \frac{1}{2}\omega^{(1)2}\} \\
&\times \int_{v_i^{(1)}}^{v_{i+1}^{(1)}} \int_{\psi_j^{(2)}}^{\psi_{j+1}^{(2)}} \frac{1}{2\pi\gamma} \exp\{-\frac{1}{2\gamma^2}\ (z^{(1)2} - 2\rho z^{(1)} z^{(2)} + z^{(2)2})\} dz^{(1)} dz^{(2)}
\end{aligned}
$$

where $v_i^{(1)} = u_i^{(1)} - \omega^{(1)}$ and $\psi_j^{(2)} = u_j^{(2)} - \rho\omega^{(1)}$, which can be expressed as

$$
\begin{aligned}
v_i^{(1)} &= \frac{\log(q_i^{(1)}) - (\mu^{(1)} + 1/2\ \sigma^{(1)2})t}{\sigma^{(1)}\sqrt{t}} \\
\psi_i^{(2)} &= \frac{\log(q_i^{(2)}) - (\mu^{(2)} - 1/2\ (\sigma^{(2)2} - 2\rho\sigma^{(1)}\sigma^{(2)}))t}{\sigma^{(2)}\sqrt{t}}
\end{aligned}
$$

Therefore

$$
J_{ij}^{(1)} = Y^{(1)}(\Phi_\rho(v_{i+1}^{(1)}, \psi_{j+1}^{(2)}) - \Phi_\rho(v_{i+1}^{(1)}, \psi_j^{(2)}) - \Phi_\rho(v_i^{(1)}, \psi_{j+1}^{(2)}) + \Phi_\rho(v_i^{(1)}, \psi_j^{(2)}))
$$

where $Y^{(1)} = \exp\{\mu^{(1)}t\}$.

Similarly,

$$
J_{ij}^{(2)} = Y^{(2)}(\Phi_\rho(\psi_{i+1}^{(1)}, v_{j+1}^{(2)}) - \Phi_\rho(\psi_{i+1}^{(1)}, v_j^{(2)}) - \Phi_\rho(\psi_i^{(1)}, v_{j+1}^{(2)}) + \Phi_\rho(\psi_i^{(1)}, v_j^{(2)}))
$$

where $Y^{(2)} = \exp\{\mu^{(2)}t\}$ and

$$
\begin{aligned}
v_i^{(2)} &= \frac{\log(q_i^{(2)}) - (\mu^{(2)} + 1/2\ \sigma^{(2)2})t}{\sigma^{(2)}\sqrt{t}} \\
\psi_i^{(1)} &= \frac{\log(q_i^{(1)}) - (\mu^{(1)} - 1/2\ (\sigma^{(1)2} - 2\rho\sigma^{(1)}\sigma^{(2)}))t}{\sigma^{(1)}\sqrt{t}}
\end{aligned}
$$

$$
\begin{aligned}
L_{ij} &= \int_{q_i^{(1)}}^{q_{i+1}^{(1)}} \int_{q_j^{(2)}}^{q_{j+1}^{(2)}} q^{(1)} q^{(2)} g(q^{(1)}, q^{(2)}) dq^{(2)} dq^{(1)} \\
&= \int_{u_i^{(1)}}^{u_{i+1}^{(1)}} \int_{u_j^{(2)}}^{u_{j+1}^{(2)}} \exp\{\omega^{(1)} y^{(1)} + \theta^{(1)}\} \exp\{\omega^{(2)} y^{(2)} + \theta^{(2)}\} \\
&\quad \times \frac{1}{2\pi\gamma} \exp\{-\frac{1}{2\gamma^2}\ (y^{(1)2} - 2\rho y^{(1)} y^{(2)} + y^{(2)2})\} dy^{(2)} dy^{(1)}
\end{aligned}
$$

Let $z^{(1)} = y^{(1)} - \omega^{(1)} - \rho\omega^{(2)}$ and $z^{(2)} = y^{(2)} - \omega^{(2)} - \rho\omega^{(1)}$, then

$$
\begin{aligned}
L_{ij} &= \exp\{\theta^{(1)} + \theta^{(2)} + \frac{1}{2}(\omega^{(1)2} + 2\rho\omega^{(1)}\omega^{(2)} + \omega^{(2)2})\} \\
&\times \int_{\xi_i^{(1)}}^{\xi_{i+1}^{(1)}} \int_{\xi_j^{(2)}}^{\xi_{j+1}^{(2)}} \frac{1}{2\pi\gamma} \exp\{-\frac{1}{2\gamma^2}\ (z^{(1)2} - 2\rho z^{(1)} z^{(2)} + z^{(2)2})\} dz^{(1)} dz^{(2)}
\end{aligned}
$$

where $\xi_i^{(1)} = u_i^{(1)} - \omega^{(1)} - \rho\omega^{(2)}$ and $\xi_j^{(2)} = u_j^{(2)} - \omega^{(2)} - \rho\omega^{(1)}$, which can be expressed as

$$\xi_i^{(k)} \quad = \quad \frac{\log(q_i^{(k)}) - (\mu^{(k)} + 1/2\ (\sigma^{(k)2} + 2\rho\sigma^{(1)}\sigma^{(2)}))t}{\sigma^{(k)}\sqrt{t}}$$

Therefore

$$L_{ij} = Y^{(12)}(\Phi_\rho(\xi_{i+1}^{(1)}, \xi_{j+1}^{(2)}) - \Phi_\rho(\xi_{i+1}^{(1)}, \xi_j^{(2)}) - \Phi_\rho(\xi_i^{(1)}, \xi_{j+1}^{(2)}) + \Phi_\rho(\xi_i^{(1)}, \xi_j^{(2)}))$$

where $Y^{(12)} = Y^{(1)}Y^{(2)} \exp\{\rho\sigma^{(1)}\sigma^{(2)}t\}$. □

Property A.26 (Existence of moments of a value function)*.* If $\mathbb{E}[r_A]$ and $\mathbb{V}[r_A]$ exist, then $\mathbb{E}[r_O]$ and $\mathbb{V}[r_O]$ exist.

Proof. Since the probability density function of asset return, $g()$, is positive everywhere, $0 \le K_i \le \int_0^\infty g(q)dq$, $0 \le J_i \le \int_0^\infty q\ g(q)dq$, and $0 \le L_i \le \int_0^\infty q^2\ g(q)dq$. By definition $\int_0^\infty g(q)dq = 1$, so for $i = 0, \ldots, n$, K_i is bounded and therefore must exist. Furthermore, if the mean and variance of the underlying asset exist, then $\int_0^\infty q\ g(q)dq$ and $\int_0^\infty q^2\ g(q)dq$ also exist and therefore J_i and L_i are bounded for $i = 0, \ldots, n$ and so must exist.

From the *Mean, variance and covariance of an option adjusted portfolio return* property we know that $\mathbb{E}[\frac{v(x)}{s}]$ and $\mathbb{V}[\frac{v(x)}{s}]$ are quadratic functions of K_i, J_i, and L_i for $i = 0, \ldots, n$ and therefore must exist. Since $\mathbb{V}[r_O] = \mathbb{V}[\frac{v(x)}{s}]$, $\mathbb{V}[r_O]$ also exists.

We have shown that any option adjusted portfolio is a combination of the risk-free asset, the risky asset, and a set of call options. We have further shown that the price of every call option exists, and therefore the price of the option strategy, p, must also exist. Since $\mathbb{E}[r_O] = \mathbb{E}[\frac{v(x)}{s}] - \frac{x_F}{s_F}.\frac{p}{s} + \frac{x_F}{s_F} - 1$, and x_F exists, and $s_F > 0$, $\mathbb{E}[r_O]$ must also exist. □

Definition A.27*.* An option adjusted portfolio (with a non-negative initial price) is *unleveraged* if the value of the portfolio at the end of the time-period is non-negative everywhere.

Property A.28 (unleveraged option adjusted portfolio condition)*.* An option adjusted portfolio is unleveraged if, and only if, $v_m = \min(v(x))$ exists and $p \le v_m/(1 + r_F) + s$.

Proof. The end value of an option adjusted portfolio is $v(x) - p(1 + r_F) + s(1 + r_F)$.

If $v(x)$ has a minimum, v_m, and if $p \le vm/(1 + r_F) + s$ then $v_m - p(1 + r_F) + s(1 + r_F) \ge 0$ and since $v(x) \ge v_m$, $v(x) - p(1 + r_F) + s(1 + r_F) \ge 0$ for all x. Therefore, the portfolio is unleveraged.

If the portfolio is unleveraged then $v(x) - p(1 + r_F) + s(1 + r_F) \ge 0$ for all x. Therefore, $v(x) \ge p(1+r_F) - s(1+r_F)$. Since p is finite, $v(x)$ must have a minimum. Furthermore $v_m - p(1 + r_F) + s(1 + r_F) \ge 0$ and therefore $v_m/(1 + r_F) + s \ge p$. □

Definition A.29*.* An unleveraged option adjusted portfolio is *well-behaved* if $v(x)/(1+r_F) + s = p$ only when $x = 0$.

Property A.30 (Existence of log moments of a value function)*.* If $\mathbb{E}[r_A]$ and $\mathbb{V}[r_A]$ exist, and $\mathbb{E}[\log(1 + r_A)]$ and $\mathbb{V}[\log(1 + r_A)]$ exist, then $\mathbb{E}[\log(1 + r_O)]$ and $\mathbb{V}[\log(1 + r_O)]$ also exist for a well-behaved unleveraged portfolio.

Proof. Recall that $r_O = \frac{v(qs)}{s} - \frac{p}{s}(1+r_F) + r_F$ where $q = \frac{x}{s}$. Since $\log(z) < z - 1$ for all $z > 0$ I can write

$$\begin{aligned}
\mathbb{E}[\log(1+r_O)] &= \int_0^\infty \log\left(\frac{v(qs)}{s} - \frac{p}{s}(1+r_F) + (1+r_F)\right) g(q)dq \\
&< \int_0^\infty \left(\frac{v(qs)}{s} - \frac{p}{s}(1+r_F) + r_F\right) g(q)dq = \mathbb{E}[R_O]
\end{aligned}$$

$$\begin{aligned}
\mathbb{E}[\log(1+r_O)^2] &= \int_0^\infty \log\left(\frac{v(qs)}{s} - \frac{p}{s}(1+r_F) + (1+r_F)\right)^2 g(q)dq \\
&< \int_0^\infty \left(\frac{v(qs)}{s} - \frac{p}{s}(1+r_F) + r_F\right)^2 g(q)dq = \mathbb{E}[R_O^2]
\end{aligned}$$

Using the *Existence of moments of a value function* property, $\mathbb{E}[R_O]$ and $\mathbb{E}[R_O^2]$ exist. Therefore, these upper bounds exist.

$$\begin{aligned}
\mathbb{E}[\log(1+r_O)] &= \int_0^\infty \log\left(v(qs) - \frac{p}{s}(1+r_F) + (1+r_F)\right) g(q)dq \\
&= \int_0^{q_1} \log\left(v(qs) - \frac{p}{s}(1+r_F) + (1+r_F)\right) g(q)dq \\
&\quad + \int_{q_1}^\infty \log\left(v(qs) - \frac{p}{s}(1+r_F) + (1+r_F)\right) g(q)dq
\end{aligned}$$

Since the portfolio is leverage and well behaved, $v(qs) - p(1+r_F) + s(1+r_F) > 0$ for all q ¿ 0 and $q_1 > 0$,

$$\log\left(v(qs) - \frac{p}{s}(1+r_F) + (1+r_F)\right) > -\infty \qquad \text{for all } q > q_1$$

and so

$$\int_0^\infty \log\left(v(qs) - \frac{p}{s}(1+r_F) + (1+r_F)\right) > -\infty$$

When $q \leq q_1$, $v(qs) = a_0 + b_0 qs$, therefore $1 + r_O = y_0 + b_0 q - \frac{p}{s}(1+r_F) + (1+r_F)$ when $q \leq q_1$ (where $y_0 = \frac{a_0}{s}$). Moreover, since $v(qs) - p(1+r_F) + s(1+r_F) \geq 0$ for all $q \geq 0$,

$$y_0 - \frac{p}{s}(1+r_F) + (1+r_F) \geq 0$$

which means $1 + r_O \geq b_0 qs$ when $q \leq q_1$. Therefore

$$\int_0^{q_1} \log\left(v(qs) - \frac{p}{s}(1+r_F) + (1+r_F)\right) g(q)dq \geq \log(b_0 s)\int_0^{q_1} g(q)dq + \int_0^{q_1} \log(q)g(q)dq.$$

Since $\mathbb{E}[\log(1+r_A)]$ exists, both of these integrals exist. Therefore

$$\int_0^{q_1} \log((v(qs) - \frac{p}{s}(1+r_F) + (1+r_F))g(q)dq > -\infty$$

Thus $\mathbb{E}[\log(1+r_O)] > -\infty$.

Finally, since $\log(x)^2 \geq 0$ for all x, $\mathbb{E}[\log(1+R_O)^2] \geq 0$.

Since $\mathbb{E}[\log(1+R_O)]$ and $\mathbb{E}[\log(1+R_O)^2]$ are bounded above and below, they must exist. Therefore $\mathbb{E}[\log(1+R_O)]$ and $\mathbb{V}[\log(1+R_O)]$ exist. □

In the remainder of this Appendix we shall be considering long-run returns of option adjusted portfolios, and will view these portfolios as a sequence of m single-period option strategies. I will let $r_{A(1)}, \ldots, r_{A(m)}$ denote the single-period returns for the underlying asset, and let $r_{O(1)}, \ldots, r_{O(m)}$ denote the single period returns for the option adjusted portfolio. The value functions of the options strategies in each single-period are denoted by $v_{(1)}(x), \ldots, v_{(m)}(x)$. Furthermore, I will let $p_{(1)}, \ldots, p_{(m)}$ denote the option adjusted portfolio prices, $s_{(1)}, \ldots, s_{(m)}$ denote the prices of the underlying asset at the start of the single periods, and $x_{(1)}, \ldots, x_{(m)}$ denote the prices of the underlying asset at the end of the single periods.

Property A.31 (Independence of option adjusted portfolio returns). If $r_{A(1)}, \ldots, r_{A(m)}$ are mutually independent random variables then $r_{O(1)}, \ldots, r_{O(m)}$ are mutually independent.

Proof. For $i = 1, \ldots, m$,

$$r_{A(i)} = \frac{x_{(i)}}{s_{(i)}} - 1 \qquad \text{and} \qquad r_{O(i)} = \frac{v_{(i)}(x_{(i)})}{s_{(i)}} - (1+r_F)\frac{p_{(i)}}{s_{(i)}} + r_F$$

Therefore

$$r_{O(i)} = \frac{v_{(i)}(s_{(i)}(1+r_{A(i)}))}{s_{(i)}} - (1+r_F)\frac{p_{(i)}}{s_{(i)}} + r_F$$

Since each $r_{O(i)}$ is a function solely of the random variable $r_{A(i)}$, if $r_{A(1)}, \ldots, r_{A(m)}$ are mutually independent then $r_{O(1)}, \ldots, r_{O(m)}$ must also be mutually independent. □

Definition A.32. Two option strategies, deployed at different points in time, are *identical* if

(1) the initial amounts in invested in the risk-free rate are the same proportions of the starting underlying asset prices; and
(2) the number of units invested in the underlying asset are the same; and
(3) the number of options are the same; and
(4) every pair of corresponding options in the two strategies has the same moneyness.

If every pair of option strategies in a sequence over time are identical, then the option adjusted portfolio is said to be *invariant*.

Property A.33 (Proportional value functions). If two option strategies are identical then their value functions, $v_{(1)}$ and $v_{(2)}$, satisfy $\frac{v_{(1)}(s_{(1)}x)}{s_{(1)}} = \frac{v_{(2)}(s_{(2)}x)}{s_{(2)}}$ for all x.

Proof. If the two strategies are identical then their value functions are expressed as

$$\begin{aligned} v_{(1)}(x) &= q_{F(1)}s_{F(1)}(1+r_F) + q_A x + \sum_{i=1}^{n} q_i \max(x - x_{(1)i}, 0) \\ v_{(2)}(x) &= q_{F(2)}s_{F(2)}(1+r_F) + q_A x + \sum_{i=1}^{n} q_i \max(x - x_{(2)i}, 0) \end{aligned}$$

Therefore

$$\begin{aligned}\frac{v_{(1)}(s_{(1)}x)}{s_{(1)}} &= q_{F(1)}\frac{s_{F(1)}}{s_{(1)}}(1+r_F)+q_A x+\sum_{i=1}^{n} q_i \max(x-\frac{x_{(1)i}}{s_{(1)}},0)\\ \frac{v_{(2)}(s_{(2)}x)}{s_{(2)}} &= q_{F(2)}\frac{s_{F(2)}}{s_{(2)}}(1+r_F)+q_A x+\sum_{i=1}^{n} q_i \max(x-\frac{x_{(2)i}}{s_{(2)}},0)\end{aligned}$$

but since they are identical, $q_{F(1)}\frac{s_{F(1)}}{s(1)} = q_{F(2)}\frac{s_{F(2)}}{s_{(2)}}$ and the same moneyness means that $\frac{x_{(1)i}}{s_{(i)}} = \frac{x_{(2)i}}{s_{(2)}}$ for $i = 1, \ldots, n$, therefore $\frac{v_{(1)}(s_{(1)}x)}{s_{(1)}} = \frac{v_{(2)}(s_{(2)}x)}{s_{(2)}}$. □

Property A.34 (Identicality of option adjusted returns). If $r_{A(1)}, \ldots, r_{A(m)}$ are identically distributed, then $r_{O(1)}, \ldots, r_{O(m)}$ are identically distributed for invariant option adjusted portfolios.

Proof. For $i = 0, \ldots, n$, and for any real-valued y, define the subset of the real numbers, $\Omega_{(i)}(y)$, as

$$\Omega_{(i)}(y) = \left\{ z : \frac{v_{(i)}(s_{(i)}z)}{s_{(i)}} - (1+r_F)\frac{p_{(i)}}{s_{(i)}} + r_F < y \right\}.$$

Since $r_{A(1)}, \ldots, r_{A(m)}$ are identically distributed, so are $1+r_{A(1)}, \ldots, 1+r_{A(m)}$. I will label $1+r_{A(i)}$ as X. Then I can write that for $i = 1, \ldots, m$

$$r_{O(i)} = \frac{v_{(i)}(s_{(i)}X)}{s_{(i)}} - (1+r_F)\frac{p_{(i)}}{s_{(i)}} + r_F$$

Therefore, $\mathbb{P}(r_{O(i)} < y) = \int_{\Omega_{(i)}(y)} \pi(x)dx$ where $\pi(x)$ is the probability density function of X. But for an invariant option adjusted portfolio, $\frac{v_{(i)}(s_{(i)}x)}{s_{(i)}} = \frac{v_{(j)}(s_{(j)}x)}{s_{(j)}}$ and the absence of arbitrage means that $\frac{p_{(i)}}{s_{(i)}} = \frac{p_{(j)}}{s_{(j)}}$ for any i and j. Therefore $\Omega_{(i)}(y) = \Omega_{(j)}(y)$ for all y and any i and j, and so $\mathbb{P}(r_{O(i)} < y) = \mathbb{P}(r_{O(j)} < y)$. Thus $r_{O(i)}$ and $r_{O(j)}$ are identical. □

Definition A.35. For a portfolio (or asset) with value $u_{(i)}$ at time i (for $i = 0, \ldots, m$), its *annualised long run return* is

$$\left(\frac{u_{(m)}}{u_{(0)}}\right)^{\frac{1}{m}}$$

Property A.36 (Chain linked returns). The annualised long-run log return of a portfolio (or asset) is the average of the single-period log returns.

Proof.

$$\left(\frac{u_{(m)}}{u_{(0)}}\right)^{\frac{1}{m}} = \left(\frac{u_{(1)}}{u_{(0)}} \times \ldots \times \frac{u_{(m)}}{u_{(m-1)}}\right)^{\frac{1}{m}} = \left((1+r_{(1)}) \times \ldots \times (1+r_{(m)})\right)^{\frac{1}{m}}$$

where $r_{(1)}, \ldots, r_{(m)}$ denote the m single-period returns.

Therefore

$$\log\left(\left(\frac{u_{(m)}}{u_{(0)}}\right)^{\frac{1}{m}}\right) = \frac{\log(1+r_{(1)}) + \ldots + \log(1+r_{(m)})}{m}$$

□

Property A.37 (Long-run option adjusted portfolio return distribution)*.* If $r_{A(1)}, \ldots, r_{A(m)}$ are independent and identically distributed, then the annualised long-run log return of an invariant option adjusted portfolio tends to a normal distribution as the number of single-periods constituting that long-run tends to infinity.

Proof. The long-run log return of the option adjusted portfolio is defined as

$$\frac{\log(1 + r_{O(1)}) + \ldots + \log(1 + r_{O(m)})}{m}$$

The two previous properties have shown that if $r_{A(1)}, \ldots, r_{A(m)}$ are independent and identically distributed, and the option adjusted portfolio is invariant, then $r_{O(1)}, \ldots, r_{O(m)}$ are independent and identically distributed. The *Existence of log moments of a value function* property says that $\mathbb{E}[\log(1 + r_O)]$ and $\mathbb{V}[\log(1 + r_O)]$ exist (I have omitted the indexing as the expectation and variance will be the same for $i = 1, \ldots, m$). Therefore, by the central limit theorem[32], in the limit as $m \to \infty$, the long-run log return is normally distributed with mean $\mathbb{E}[log(1 + r_O)]$ and variance $\mathbb{V}[\log(1 + r_O)]/m$. □

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