Dynamic Allocation Decisions in the Presence of Funding Ratio Constraints

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Abstract

The recent pension crisis has triggered a fierce debate in most developed countries between advocates of a tighter regulation designed to provide explicit incentives for pension funds to increase their focus on risk management, and those arguing that imposing short-term funding constraints and solvency requirements to such long-term investors would only increase the cost of pension financing. We analyze this question in the context of a continuous-time dynamic asset allocation model for an investor facing liability commitments subject to inflation and interest rate risks. In the presence of funding ratio constraints, the optimal policy is shown to involve dynamic allocation strategies that are reminiscent of portfolio insurance strategies, extended to an asset-liability management context. We also show that the introduction of maximum funding ratio targets would allow pension funds to decrease the cost of downside liability risk protection while giving up part of the upside potential beyond levels where marginal utility of wealth (relative to liabilities) is low or almost zero. Overall, our results suggest that it is not so much the presence of short-term funding ratio constraints that is costly per se for pension funds, but rather their reluctance to implement risk-management strategies that are optimal given such regulatory constraints.

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1 Introduction

Market difficulties at the turn of the millennium have drawn attention to risk management practices of institutional investors in general and defined benefit pension plans in particular. What has been labeled as a "perfect storm of adverse market conditions" has devastated many corporate pension plans, with negative equity market returns that have eroded plan assets at the same time as declining interest rates have increased market-to-market value of benefit obligations. In 2003, the defined benefit pension plans for the companies included in the S&P 500 and the FTSE 100 index faced a cumulative deficit estimated at \$225 billion and £55 billion, respectively, while the worldwide deficit reached an estimated \$1,500 to \$2,000 billion.[1] To better understand the magnitude of the crisis, and the scale and rapidity of the deterioration in pension funding status, it is perhaps worth noting that in the United States, defined benefits plans from S&P500 companies were enjoying a total *surplus* of \$239 billion at the end of 1999, a mere 3 years earlier![2].

That pension funds have been so dramatically affected by market downturns has raised with heightened scrutiny the question of risk management for pension funds, and many have argued that their asset allocation policies have not been entirely consistent with a sound liability risk hedging process. In particular, it has been argued that asset allocation decisions of defined benefits pension schemes were often too heavily skewed towards equities in the absence of any protection with respect to their downside risk relative to liabilities. According to an annual survey, it turns out that by 1992, percentage holdings in equities by pension funds were 75% in the UK, 47% in the US, 18% in the Netherlands and 13% in Switzerland. In 2001, midway through the bear market, pension funds had 64% of their total assets in equities in the UK, 60% in the US, 50% in the Netherlands, and 39% in Switzerland.[3] As a result of such a domination of equities, the increase in liability value that followed decrease in interest rates was only partially offset by the parallel increase in the value of the bond portfolio.

From an academic perspective, a number of papers have focused on extending Merton's intertemporal selection analysis (see Merton (1971)) to account for the presence of liability constraints in the asset allocation policy. Broadly speaking, asset-liability management (ALM) distinguishes itself from pure asset management by the fact that what matters from an ALM perspective is not total terminal wealth, but how terminal wealth compares to terminal value of future liabilities. More precisely, the presence of future liability commitments is accounted for by a focus on terminal wealth net of the value at horizon of future liability payments, a quantity known as the pension fund *surplus (*or as the pension fund *deficit* when it is

[1]See Standard Life Investments (2003) and Watson Wyatt (2003).
[2]See Standard Life Investments (2003).
[3]See the Lane, Clark & Peacock Actuaries and Consultants (2003).

negative).[4] A first step towards the application of optimal portfolio selection theory to the problem of pension funds has been taken by Merton (1993) himself, who studies the allocation decision of a university that manages an endowment fund. In a similar spirit, Rudolf and Ziemba (2004) have formulated a continuous-time dynamic programming model of pension fund management in the presence of a time-varying opportunity set, where state variables are interpreted as currency rates that affect the value of the pension's asset portfolio. Also related is a paper by Sundaresan and Zapatero (1996), which is specifically aimed at asset allocation and retirement decisions in the case of a pension fund. In a recent paper, van Binsbergen and Brandt (2007) complement this early work by analyzing how various regulatory rules with respect to the valuation of liabilities impact optimal investment decisions.

In a nutshell, the main insight from this strand of the literature is that the presence of liability risk induces the introduction of a specific hedging demand component in the optimal allocation strategy, as typical in intertemporal allocation decisions in the presence of stochastic state variables. These results suggest that very conservative pension fund would optimally invest a significant fraction of their assets in the *liability-hedging portfolio*, mostly invested in nominal or inflation-indexed bonds, so as to minimize as much as possible the probability and magnitude of a shortfall. As a consequence, they also suggest that high levels of investment in equities before the start of the bear market might have indeed been the primary cause of the pension crisis.

In an attempt to provide direct incentives for pension funds to increase the focus on risk management, and in view of protecting the interests of beneficiaries, formal funding ratio constraints have been introduced by the regulator in most developed countries. This issue has given rise to a fierce debate between advocates of a tighter regulation, not only in the US but also in Europe, and those arguing that it would only result in a severe welfare loss.[5] The introduction of funding ratio constraints has been particularly criticized by a number of experts, who find that imposing such short-term constraints to long-term investors could be counter-productive. For example, in a report prepared on pension funding rules for the OECD, Pugh (2003) makes the following argument. "Minimum funding standards in many countries are designed around (...) current market yields on long-term bonds. In order to avoid problems, especially in jurisdictions that require immediate correction of the (perceived) underfunding, a plan sponsor is tempted to over-invest in such long-term bonds. (...) However, pension plans in the long term (...) need substantial investments in equities. Otherwise, the investments may be inefficient, and the cost of the pension plan to the plan sponsor will

[4] In fact, defined benefit pension commitments represent a short position in collateralized defaultable bonds issued by the sponsor company and privately held by employees, where the assets of the pension plans are the collateral, in exchange for which the company receives the present value of lower wage demands.

[5] In Europe it has even been proposed that the Solvency II regulatory framework, originally designed for insurance companies, should also be applied to pension funds.

therefore increase. (...) In the area of minimum funding, as with other areas of legislation, there is a fine line between (over)protecting the interests of DB plan members and destroying the incentives for employers to sponsor such plans."

In fact, there are two main (related) arguments that are put forwards by advocates of looser regulations on pension funds. The first argument is related to the *cost of short-termism*. In the presence of short–term funding ratio constraints, it is required that the sponsor company makes an additional contribution so as to bring the funding ratio back to minimum required value when needed. Consider for example a very tight regulation that enforces funding ratio constraints every year, and require immediate funding of any deficit with respect to the liabilities. Such "short-termism" of the regulation de facto prevents pension funds to benefit from possible recoveries due to market performance. In other words, when a pension fund is under-funded at a given date, there is a non-zero probability that it will get back to a fully funding status (or better) at a later horizon date without any additional contribution, simply because of solid market performance. When the regulation imposes that funding ratio constraints ought to be satisfied every year, however, the sponsor company has to make such additional contributions, even though they eventually would prove to be unnecessary in those states of the world, for which the pension fund will as a consequence end up with unnecessarily high funding levels. The second argument is precisely related to *contribution irreversibility*. While sponsor companies are required to make additional contributions when needed to bring asset value back to minimum regulatory levels, it is typically not possible for them to obtain refunds in those states of the world where funding ratios are very high. In the end, excess fund levels typically result in improved benefits to pensioners, and it is only in exceptional circumstances that shareholders of sponsor companies can enjoy to extract some of the pension fund excess of assets. These two arguments have been summarized by Pugh (2003) in the following statement: "In many countries, the minimum funding standards focus on the pension fund assets exceeding the pension plan's accrued liabilities on every measurement date. (...) If an asset/liability type of minimum funding measure is to be introduced or retained, then legislation should not require the immediate and complete correction of any underfunding that the test purports to reveal. Asset values fluctuate, and funding shortfalls may disappear as quickly as they had appeared. It is counterproductive for a plan sponsor to make high additional contributions and then find, one or two years later, that the markets have recovered and the plan now has an embarrassing funding excess."

While these arguments seem to make compelling intuitive sense, a formal analysis of the welfare loss, if any, related to regulatory short-termism and contribution irreversibility, has yet to be performed. This paper extends the afore-mentioned literature on asset allocation decisions with liability commitments by analyzing the impact of funding ratio constraints in the context of a continuous-time model for intertemporal allocation decisions, and revisit the question of the costs and benefits of risk management from an asset-liability management perspective. Given that interest rate and inflation uncertainty are the two main risk factors impacting pension liability values, we cast the problem in a setting with stochastic interest and inflation rates. Using the martingale approach to portfolio optimization problems, we provide explicit solutions in the unconstrained case, as well as when funding ratio constraints are explicitly or implicitly imposed. In the unconstrained case, we confirm that the optimal strategy involves a fund separation theorem that legitimates investing in a liability-hedging portfolio, in addition to the standard performance-seeking portfolio (speculative demand). When funding ratio constraints are introduced, optimal policies, for which we obtain analytical expressions, are shown to involve a dynamic allocation to the performance-seeking portfolio that is a function of the margin for error measured in terms of the distance between the current asset value and the minimum level consistent with the funding ratio constraints. These strategies, which fall within the category of risk-controlled dynamic asset allocation strategies, are reminiscent of Constant Proportion Portfolio Insurance (CPPI) or Option-Based Portfolio Insurance (OBPI) strategies, which they extend to a relative (with respect to liabilities) risk context.[6] We also show that the introduction of maximum funding ratio targets would allow pension funds to decrease the cost of downside liability risk protection while giving up part of the upside potential beyond levels where marginal utility of wealth (relative to liabilities) is low or almost zero. Finally, we introduce an *irrelevance principle* (see Proposition 8), which states that following an unconstrained strategy and adding contributions to make up for the deficit, if any, with respect to the minimum funding ratio requirement leads to the same payoff as following a constrained strategy when contributions are reversible and funding ratio constraints are only imposed at the terminal date. As a corollary, this result suggests that risk management should allow for welfare gains in the presence of contribution irreversibility and regulatory short-termism. To provide a numerical assessment of these effects, we implement various optimal risk-controlled strategies in the context of a typical liability payment schedule of an actual pension fund. For reasonable parameter values, these strategies are found to allow for a significant access to the upside potential of the performance-seeking portfolio, coupled with downside protection relative to the liability-based benchmark. We confirm that risk management adds value in the presence of contribution irreversibility and regulatory short-termism, and we find that the induced welfare gains are very significant. We also compare an unconstrained strategy with additional contributions required every year in case of a deficit (regulation with short-term constraint) to an unconstrained strategy with additional contributions only at horizon (regulation with long-term constraint), and we provide a formal quantitative measure for the cost of short-termism.

These results have a number of important implications in the perspective of the current

[6]CPPI strategies have originally been introduced by Black and Jones (1987) and Black and Perold (1992).

debate regarding pension fund regulation. In particular they suggest that tight regulations, with short-term funding ratio constraints, do not involve significant welfare losses and additional burdens in terms of additional contributions, especially when marginal utility decreases sharply beyond a given threshold. They also suggest that what is actually costly for long-term investors is not so much the presence of funding ratio constraints per se, but rather the reluctance for such investors to follow risk-controlled strategies that are optimal given these regulatory funding ratio constraints. While the focus of the paper is mostly on defined benefits pension funds, we believe that some of the insights we obtain in this paper apply more generally beyond this particular context, and could shed some light on the benefits and costs of risk management strategies for defined contribution pension plans, insurance companies, or even households, who also face a number of implicit or explicit liabilities.

In addition to the literature on asset-liability management, our paper is also strongly related to the literature on portfolio insurance and more generally on portfolio decisions with minimum target terminal wealth. This literature has evolved according to at least two main directions. The first strand of the literature, starting with Leland (1980) and extended by Benninga and Blume (1985) or Franke et al. (1998), approaches the question from a positive angle: taking as given a set of standard convex payoffs, these papers examine the features of investors' preferences and market characteristics that would support a rational non-zero holding of these derivatives contracts. They have mostly found that only severe forms of market incompleteness and/or the presence of background risk can justify holding such payoffs. In a related effort, but moving beyond the paradigm of expected utility maximization, Driessen and Maenhout (2007) have shown that it is only with highly distorted probability assessments that one can obtain positive portfolio weights for puts (cumulative prospect theory and anticipated utility) and straddles (anticipated utility). The second strand of the literature, initiated by Brennan and Solanki (1981), has examined the question from a more normative angle by searching for the design of optimal payoffs from the investor's standpoint. To this strand of the literature are related papers on portfolio allocation with wealth constraints, including Grossman and Vila (1989), Cox and Huang (1989), Basak (1995), Basak (2002) or Grossman and Zhou (1996). These papers rationalize constant proportion portfolio insurance or option-based portfolio insurance by showing that such strategies are optimal in the presence of wealth constraints. Our paper extends this latter strand of the literature by revisiting in the presence of liability commitments the question of optimal design of non-linear payoffs and the related question of the costs and benefits of risk management. Finally, because of its focus on asset allocation decisions with a liability benchmark, our paper is also related to the literature on dynamic asset allocation models with performance benchmarks. Single-agent portfolio allocation models with benchmark constraints include notably Browne (2000) in a complete market setting, or Teplá (2001) who also includes constraints on relative performance. Another formally related paper is Brennan and Xia (2002) who study in an incomplete market setting asset allocation decisions when an inflation index is used as a numeraire.[7]

The rest of the paper is organized as follows. In section 2, we introduce a formal continuous-time model of dynamic asset allocations decisions in the presence of liability commitments. In section 3, we provide a numerical analysis of the cost of regulatory short-termism. In section 4 and section 5, we analyze the impact of formal minimum and/or maximum funding ratio constraints on optimal allocation strategies. In section 6, we provide a numerical measure of the welfare cost induced by the reluctance of following risk-controlled strategies that are optimal given regulatory constraints. Section 7 concludes and presents suggestions for further research. Technical details and proofs of the main results are relegated to dedicated appendices.

2 A Formal Model of Asset-Liability Management

In this section, we introduce a stylized continuous-time asset allocation model for a pension fund facing liability constraints.

2.1 Stochastic Model for State Variables and Risky Assets

We let $[0, T_0]$ denote the (finite) time span of the economy, where uncertainty is described through a standard probability space $(\Omega, \mathscr{A}, \mathbb{P})$. In what follows, T_0 can be thought of as the date of the last pension payment by a pension fund (after which the pension fund will be terminated), to be distinguished from the investment horizon, which can be some arbitrary date denoted by $T \leq T_0$. We assume that financial markets are frictionless.

Regarding the liability side, inflation risk and interest rate risk appear as the two most relevant risk factors. This is because pension benefits are typically inflation-indexed, and the typically long duration of liability payments make their current value highly sensitive to changes in interest rates. In what follows, we model the nominal short-term interest rate as an Ornstein-Uhlenbeck process (see. Vasicek (1977)) and the price index as a Geometric Brownian motion:

$$\mathrm{d}r_t = a(b - r_t)\,\mathrm{d}t + \overline{\sigma}_r\,\mathrm{d}\overline{z}_t^r$$
$$\frac{\mathrm{d}\Phi_t}{\Phi_t} = \varphi\,\mathrm{d}t + \overline{\sigma}_\Phi\,\mathrm{d}\overline{z}_t^\Phi$$

where $\overline{z}^r$ and $\overline{z}^\Phi$ follow standard correlated Wiener processes under $\mathbb{P}$. φ represents the in-

[7]Equilibrium implications of the presence of performance benchmarks are discussed in Cuoco and Kaniel (2001), Gómez and Zapatero (2003), or Basak et al. (2006).

stantaneous expected inflation rate, which we assume to be constant for simplicity.[8]

On the asset side, we assume that the menu of asset classes includes a unit zero-coupon bond with payoff 1 at maturity τ_1 and a price $B(t, \tau_1)$ at time t. So as to stay within a complete market environment, we also assume that inflation risk is spanned by an inflation-indexed unit zero-coupon bond of maturity τ_2, i.e., a bond with payoff given by Φ_{τ_2}. The price at time t of the inflation-linked bond is denoted by $I(t, \tau_2)$, which will be made explicit in proposition 1 below. Moreover, we assume that there exists one stock whose price S_t evolves as:

$$\mathrm{d}S_t = S_t[\mu_S \,\mathrm{d}t + \sigma_S \,\mathrm{d}z_t^S]$$

The dynamics of these state variables can be rewritten in vector form as:

$$\begin{aligned} \mathrm{d}r_t &= a(b - r_t)\,\mathrm{d}t + \boldsymbol{\sigma}_r' \,\mathrm{d}\boldsymbol{z}_t \\ \mathrm{d}\Phi_t &= \Phi_t[\varphi \,\mathrm{d}t + \boldsymbol{\sigma}_\Phi' \,\mathrm{d}\boldsymbol{z}_t] \\ \mathrm{d}S_t &= S_t[\mu_S \,\mathrm{d}t + \boldsymbol{\sigma}_S' \,\mathrm{d}z_t] \end{aligned} \tag{2.1}$$

where $\boldsymbol{z}$ is a 3-dimensional Wiener process. Throughout the paper we assume that the information available to the investor at time t is $\mathscr{F}_t$, the augmented sigma-field generated by $\boldsymbol{z}$ up to time t.

Assuming a complete financial market, there exists a unique price of risk vector $\boldsymbol{\lambda}$, which we assume to be constant. This vector $\boldsymbol{\lambda}$ defines a unique equivalent martingale measure by:

$$\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}} = \exp\left[-\int_0^{T_0} \boldsymbol{\lambda}_s' \,\mathrm{d}\boldsymbol{z}_s - \frac{1}{2}\int_0^{T_0} \|\boldsymbol{\lambda}_s\|^2 \,\mathrm{d}s\right] \tag{2.2}$$

and a unique pricing kernel process M through:

$$M_t = \exp\left(-\int_0^t r_s \,\mathrm{d}s\right) \mathbb{E}_t\left[\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\right]$$

It then follows from Girsanov's theorem that the process $\hat{\boldsymbol{z}}$ defined by $\mathrm{d}\hat{\boldsymbol{z}}_t = \mathrm{d}\boldsymbol{z}_t + \boldsymbol{\lambda}_t \,\mathrm{d}t$ is a $\mathbb{Q}$-Brownian motion. We are now able to write the prices of the nominal and indexed zero-coupon bonds that are available for trading.

Proposition 1 *The prices of the nominal and of the real zero-coupon bonds of respective maturities τ_1 and τ_2 are given by:*

$$B(t, \tau_1) = e^{\alpha(\tau_1 - t) r_t + \beta_1(\tau_1 - t)} \quad \textit{and} \quad I(t, \tau_2) = \Phi_t e^{\alpha(\tau_2 - t) r_t + \beta_2(\tau_2 - t)}$$

[8] Brennan and Xia (2002) and Munk et al. (2004) assume that the expected inflation rate follows an Ornstein-Uhlenbeck process.

where:

$$\alpha(s) = -\frac{1-e^{-as}}{a}, \quad \widetilde{b} = b - \frac{\boldsymbol{\sigma}_r'\boldsymbol{\lambda}}{a}, \quad \widetilde{\varphi} = \varphi - \boldsymbol{\sigma}_\Phi'\boldsymbol{\lambda}$$
$$\beta_1(s) = -\widetilde{b}s + \widetilde{b}\frac{1-e^{-as}}{a} + \frac{\|\boldsymbol{\sigma}_r\|^2}{2a^2}\left[s - 2\frac{1-e^{-as}}{a} + \frac{1-e^{-2as}}{2a}\right]$$
$$\beta_2(s) = \left(\widetilde{\varphi} - \frac{\|\boldsymbol{\sigma}_\Phi\|^2}{2} - \widetilde{b}\right)s + \widetilde{b}\frac{1-e^{-as}}{a} + \frac{1}{2}\int_0^s \left\|\frac{1-e^{-au}}{a}\boldsymbol{\sigma}_r - \boldsymbol{\sigma}_\Phi\right\|^2 du$$

In particular, the volatility vectors of $B(\cdot,\tau_1)$ and $I(\cdot,\tau_2)$ are given by:

$$\boldsymbol{\sigma}_B(t,\tau_1) = \alpha(\tau_1 - t)\boldsymbol{\sigma}_r \quad \textit{and} \quad \boldsymbol{\sigma}_I(t,\tau_2) = \alpha(\tau_2 - t)\boldsymbol{\sigma}_r + \boldsymbol{\sigma}_\Phi$$

Proof. The expression for the nominal zero-coupon bond is standard. See e.g. Vasicek (1977). The expression for the inflation-linked zero-coupon bond follows from the equalities:

$$\Phi_{\tau_2} = \Phi_t \exp\left[\int_t^{\tau_2}\left(\widetilde{\varphi} - \frac{\|\boldsymbol{\sigma}_\Phi\|^2}{2}\right)\mathrm{d}u + \int_t^{\tau_2}\boldsymbol{\sigma}_\Phi'\,\mathrm{d}\hat{\boldsymbol{z}}_u\right]$$
$$\int_t^{\tau} r_u\,\mathrm{d}u = \widetilde{b}\left[\tau_2 - t - \frac{1-e^{-a(\tau_2-t)}}{a}\right] + \frac{1-e^{-a(\tau_2-t)}}{a}r_t + \frac{1}{a}\int_t^{\tau_2}\left(1-e^{-a(\tau_2-u)}\right)\boldsymbol{\sigma}_r'\,\mathrm{d}\hat{\boldsymbol{z}}_u$$

and from standard computations of expectations of log-normal random variables. ■

2.2 Net Wealth Process

We consider a pension fund managing financial assets and paying a stream of pension payments. For simplicity, we do not model the stream of contributions from the sponsor company, and assume instead that it can be summarized by an initial endowment A_0 to the pension fund.[9] This initial wealth can be invested in the stock, the two zero-coupon bonds, and the cash.

We denote with $\boldsymbol{\omega}_t$ the vector of weights describing the portfolio at time t, and with $\boldsymbol{\sigma}_t$ the volatility matrix of S, $B(\cdot,\tau_1)$ and $I(\cdot,\tau_2)$ at time t. The value of the financial portfolio, A, evolve as:

$$\mathrm{d}A_t = A_t[r_t + \boldsymbol{\omega}_t'\boldsymbol{\sigma}_t'\boldsymbol{\lambda}]\,\mathrm{d}t + A_t\boldsymbol{\omega}_t'\boldsymbol{\sigma}_t'\,\mathrm{d}\boldsymbol{z}_t - \mathrm{d}V_t \tag{2.3}$$

where $\mathrm{d}V_t$ is the payment to pensioners between dates t and $t+\mathrm{d}t$. This representation can accommodate continuous payments as well as lump-sum payments at dates t_1, ..., t_n. In this latter case, $\mathrm{d}V_t$ should be formally written as $\mathrm{d}V_t = \sum_{i=1}^n l_{t_i}\,\mathrm{d}H_t^{t_i}$, where H^{t_i} is an Heaviside function, $H_t^{t_i} = \mathbb{1}_{\{t \geq t_i\}}$. In what follows, we shall sometimes consider a specific case of the

[9]In a more general setting, one should allow for an endogenous contribution policy from a sponsor company perspective (see suggestions for further research in the conclusion). Note that we consider in section 6 a specific kind of contribution strategy aiming at covering the funding gap when needed.

discrete payment model, where a single payment takes place, at time T_0, a situation we shall refer to as the *zero-coupon case*. The *generic case* will be the generic situation, where the continuous or discrete nature of the payments is not specified.

Since the financial market is complete, the stream of future payments can be valued as the dividend flow of a financial asset. Hence the quantity

$$L_t^{T_1,T_2} = \mathbb{E}_t^{\mathbb{Q}}\left[\int_{]T_1,T_2]} e^{-\int_t^s r_u \, \mathrm{d}u} \, \mathrm{d}V_s\right], \quad t \leq T_1 < T_2 \tag{2.4}$$

is the price that an agent would have to pay at time t to receive the payment stream $\mathrm{d}V$ from date T_1 excluded to date T_2 included.[10] We will also let: $L_t = L_t^{t,T_0}$ denote the total liability value, i.e., the discounted value of all future liability payments, at date t. With these notations, the budget constraint (see proposition 2.2 in Cox and Huang (1988)) becomes:

$$A_t = \mathbb{E}_t\left[\frac{M_s}{M_t}A_s\right] + L_t^{t,s}, \quad t < s \tag{2.5}$$

Throughout the paper we take $l_t = n_t\Phi_t$, where n is a *nonnegative deterministic* function of time representing the size of the population to which benefits will be provided for. It is straightforward to show that for $T_1 < T_2$, $L_t^{T_1,T_2}$ is a function $\mathscr{L}^{T_1,T_2}$ of (t, r_t, Φ_t). Ito's lemma then gives the volatility vector of L^{T_1,T_2}:

$$\boldsymbol{\kappa}_t^{T_1,T_2} = \frac{\mathscr{L}_r^{T_1,T_2}\boldsymbol{\sigma}_r + \mathscr{L}_\Phi^{T_1,T_2}\Phi_t\boldsymbol{\sigma}_\Phi}{L_t^{T_1,T_2}} \tag{2.6}$$

When $T_1 = t$ and $T_2 = T_0$, we simply write $\boldsymbol{\kappa}_t$ rather than $\boldsymbol{\kappa}_t^{t,T_0}$.

In the zero-coupon case, it is assumed that the pension fund makes a single payment, at time T_0. We thus have $L_t^{T,T_0} = n_{T_0}I(t,T_0)$ for any $T \in [t, T_0[$ and the volatility vector of L is $\boldsymbol{\kappa}_t = \boldsymbol{\sigma}_I(t,T_0)$. In particular, the volatility vector is deterministic in the zero-coupon case (while it is stochastic when a stream of continuous- or discrete- liability payments are considered). This property allows for explicit pricing of the terminal optimal net wealth, and the associated dynamic asset allocation strategy, as will be made clear below.

2.3 Objectives and Optimal Asset Allocation Decisions

There are various stakeholders involved in an ALM problem, including most notably the shareholders of the sponsor company and the beneficiaries of the pension plan (workers and pensioners). The objective of this paper is not to analyze the conflicts of interest that arise between

[10] In this paper, we assume away the presence of default risk on liability payments (see section 7 for a discussion).

these agents, but instead to focus on the perspective of the pension fund manager, an agent who acts on the behalf of the two afore-mentioned stakeholders. The objective of this paper is to adopt the perspective of the pension fund manager, an agent who acts on the behalf of the shareholders and workers of the company. Preferences of the manager are expressed here on the terminal funding ratio F_T, where:

$$F_t \equiv \frac{A_t}{L_t}, \quad t < T_0 \tag{2.7}$$

From an interpretation standpoint, focusing on the funding ratio amounts to using the liability value process $(L_t)_{t\geq 0}$, as opposed to the bank account, as a numeraire, an approach that has been used for example by van Binsbergen and Brandt (2007), and which we also adopt in this paper.[11] The reduced-form program assumed for the pension fund manager is thus:

$$\max_{\boldsymbol{\omega}} \mathbb{E}[u(F_T)] \tag{2.8}$$

where we assume that the pension fund manager has CRRA preferences, with relative risk aversion γ:[12]

$$u(x) = \begin{cases} \dfrac{x^{1-\gamma}}{1-\gamma} & \text{for } x > 0 \\ -\infty & \text{for } x \leq 0 \end{cases}$$

The following proposition presents the expression for the optimal policy and optimal asset value process for program (2.8) in the zero-coupon case.[13] To obtain this solution, we use the martingale approach in complete markets developed by Cox and Huang (1989). Details of derivation are relegated to appendices.

Proposition 2 • *The optimal payoff in* (2.8) *in the generic case is:*

$$A_T^{*u} = \frac{A_0}{\mathbb{E}\left[(M_T \Phi_T)^{1-\frac{1}{\gamma}}\right]} M_T^{-\frac{1}{\gamma}} L_T^{1-\frac{1}{\gamma}}$$

• *In the zero-coupon case, the optimal strategy reads: is given by the following asset strat-*

[11] One natural alternative would consist in solving the optimal allocation problem when preferences are expressed in terms of the surplus $A_T - L_T$ as opposed to the funding ratio A_T/L_T. This would generate in fact, and quite unsurprisingly, very similar results (an overview of the results when preferences are expressed over the surplus can be obtained from the authors upon request).

[12] Detemple and Rindisbacher (2008) consider a general class of utility functions, encompassing the CRRA case.

[13] Solutions to the more general case with multiple liability payments can be obtained from the authors upon request.

egy:

$$\boldsymbol{\omega}_t^{*u} = \frac{\boldsymbol{e}'_{n+2}\boldsymbol{\sigma}_t^{-1}\boldsymbol{\lambda}}{\gamma}\boldsymbol{\omega}_t^{PSP} + [\alpha(T_0 - t)\boldsymbol{e}'_{n+2}\boldsymbol{\sigma}_t^{-1}\boldsymbol{\sigma}_r + \boldsymbol{e}'_{n+2}\boldsymbol{\sigma}_t^{-1}\boldsymbol{\sigma}_\Phi]\left(1 - \frac{1}{\gamma}\right)\boldsymbol{\omega}_t^{LMP}(T, T_0)$$

where:

$$\boldsymbol{\omega}_t^{PSP} \equiv \frac{\boldsymbol{\sigma}_t^{-1}\boldsymbol{\lambda}}{\boldsymbol{e}'_{n+2}\boldsymbol{\sigma}_t^{-1}\boldsymbol{\lambda}}, \qquad \boldsymbol{\omega}_t^{LMP}(T, T_0) = \frac{\boldsymbol{\sigma}_t^{-1}\boldsymbol{\sigma}_I(t, T_0)}{\boldsymbol{e}'_{n+2}\boldsymbol{\sigma}_t^{-1}\boldsymbol{\sigma}_I(t, T_0)}$$

We find that the solution involves the standard performance-seeking portfolio (PSP) and a liability-hedging or liability-matching portfolio (LMP). This portfolio has the following property, which is typical of intertemporal hedging demand terms in dynamic asset allocation models (see Merton (1973)): $\boldsymbol{\omega}_t^{LMP}(T, T_0)$ maximizes the correlation between the returns on the asset portfolio and the return on the present value of future pension payment. In fact, in this complete market setting, the maximum correlation achieved is equal to 1. In case the maturity of the inflation-linked bond coincides with the date of the unique payment T_0, the liability-matching portfolio is fully invested in this inflation-linked bond; otherwise it involves the combination of cash and the inflation-linked-bond needed for reaching the target duration. It should be noted that the optimal portfolio strategy does not involve a *separate* interest rate hedging component. While interest rate risk impacts the asset value, it also impacts liability value in such a way that the net impact at the funding ratio level is trivial.

3 Measuring the Cost of Regulatory Short-Termism

We now turn to an empirical testing of the optimal strategies discussed in the previous section. To this end, we use a schedule of liability payments provided for by a Dutch pension fund and displayed in table 1, from which we obtain that the date of the last scheduled payment is $T_0 = 75$ years. These cash-flows represent *real* expected pension payments, to which a cumulative inflation factor should be added so as to obtain the actual liability payment.[14] The duration of the pension fund liability is the maturity of the indexed zero-coupon bond that has the same sensitivity to interest rates as the coupon bond that models the liability. Since the present value at time 0 of all future payments is equal to L_0, the duration τ_0 is defined by:

$$-\frac{1 - e^{-a\tau_0}}{a} = \frac{1}{L_0}\frac{\partial L_0}{\partial r_0} = \frac{1}{L_0}\sum_{t_i=1}^{75} n_{t_i} I(0, t_i) - \frac{1 - e^{-at_i}}{a} \tag{3.1}$$

[14]In practice, inflation indexation is sometimes conditional, with indexation conditions that can be complex and typically depend on the funding ratio of the pension fund and the inflation rate, combined with a minimum and maximum level of indexation. We shall assume away this additional complexity in the empirical exercise that follows, and consider for simplicity a full indexation payment.

Numerically, we get that $\tau_0 = 11.32$ years.

TABLE 1 ABOUT HERE

Our goal in this section is not only to illustrate the empirical properties of the dynamic allocation strategy in the presence of liability constraints, but also to provide a formal measure of the cost of regulatory short-termism.

3.1 Base Case Empirical Exercise

This subsection serves the purpose of providing an illustration and empirical testing of the various strategies presented in the previous section, based on actual pension fund data. With no loss of generality, we assume that the investment opportunity set includes a single stock index, regarded as an efficient combination of individual stocks, in addition to a zero-coupon bond and an inflation-indexed bond with maturity corresponding to the duration of pension payments. Our base case parameters are taken from Munk et al. (2004), who also model the nominal interest rate as an Ornstein-Uhlenbeck process and the price index as a Geometric Brownian motion.[15] Table 2 summarizes our base case set of parameter values.

TABLE 2 ABOUT HERE

For each set of parameters values, $N = 5000$ optimal terminal asset values are obtained. In all cases, we have assumed that the pension fund was initially fully funded, that is $A_0 = L_0$.

In table 3, we provide information regarding the distribution of the final funding ratio when no constraint is introduced. As expected, we find that the dispersion of the final funding ratio increases with T and decreases with γ. Indeed, a lower risk-aversion parameter implies a higher investment in the performance-seeking portfolio, and hence a higher performance potential coupled with a higher funding risk. On the other hand, for a given risk-aversion parameter value, we find that the range of funding ratio values increases with the time-horizon T as more time is allowed for uncertainty to play a role. Even for $\gamma = 10$, we find that the minimum funding ratio obtained is lower than 90% for a one-year horizon, and lower than 70% for a 50 year horizon. This provides justification for the introduction of funding ratio constraints aiming at imposing a left-truncation of the final funding ratio distribution.

TABLE 3 ABOUT HERE

[15] The main difference between their model and ours is that they also assume an Ornstein-Uhlenbeck process for the expected inflation rate, whereas we take it as a constant, which we assume is given by the long-term mean level used by Munk et al. (2004). Beside, Munk et al. (2004) provide an estimate for the market price of interest rate risk, but neither for the market of inflation risk nor for the market price of equity risk. We set the former at 0, and we set the latter to the value found used in Brennan and Xia (2002), namely 0.343.

3.2 Additional Contributions Induced by Regulatory Short-Termism

In this subsection, we test for the impact of short-term funding ratio constraints in terms of additional contributions. As outlined in the introduction, there are in fact two aspects in the short-termism of regulatory constraints. The first aspect relates to the frequency of regulatory actuarial valuations, which typically take place every year (Belgium, Brazil, Netherlands, US) or every 3 years (Canada, UK, Portugal, Switzerland or Ireland). The second aspect relates to how much time is allowed for recovery plans, that is how many years the sponsor company has to fund the deficit measured at time t. This recovery time also varies across countries and regulatory environments. In rare instances, immediate funding is required, as in Germany when the funding ratio is below 100%. Most often, some recovery time is allowed for. This recovery time is equal to 3 years in Ireland and the Netherlands, versus 5 years in the US and in Canada (where an additional margin is allowed for). In other countries, the regulation does not impose any general recovery time, so that recovery time becomes plan specific; this is the case for Belgium, the UK (where it reaches 7.5 years on average), or Portugal (between 3 years to 10 years), where sponsor companies are expected to provide the regulator with a reasonable recovery plan indicating how they expect to manage to close to the funding gap, and how much time they expect to take to do so.

Given that an analysis of the impact of different regulatory environments is obviously of high practical importance, we formalize this question as follows. We consider a pension fund with horizon T and a single liability payment at date $T_0 > T$. The pension fund is assumed to follow the optimal unconstrained strategy, as written in Proposition 2. The regulator imposes that the funding ratio should be checked every δt years, where typical values of δt are 1 or 3, as recalled earlier. For $i = 1, ... \frac{T}{\delta t}$, we denote by $s_i = i\delta t$ the date of the i^{th} actuarial valuation. At every measurement date, the sponsor is called to contribute if the funding ratio lies below the required minimum level k. The net asset of the pension fund just after the contribution is thus equal to the net asset just before, plus the contribution. The level of the contribution is equal to the gap between the net asset and the minimum asset imposed by regulatory constraint, but the sponsor company may split its contributions into m parts, consistent with a m years duration allowed for recovery. In other words, we take the actual contribution at time $s_i < T$ is:

$$\text{Cont}_{s_i} = \frac{1}{m} \left(kL_{s_i} - A_{s_i -} \right)^+ \tag{3.2}$$

with $m = 1$ describing a stringent regulation where the sponsor is required to close the whole funding gap immediately. For all m values, we assume that the sponsor has to finance any remaining deficit immediately at the horizon date T. The present value of the stream of contributions reads:

$$C_0(\delta t, m, A_0) = \sum_{i=1}^{\frac{T}{\delta t}} \mathbb{E}^{\mathbb{Q}} \left[\exp \left(- \int_0^{s_i} r_u \, \mathrm{d}u \right) \mathrm{Cont}_{s_i} \right]$$

where we have emphasized the dependence upon δt, m and A_0.

3.3 A Formal Measure of the Cost of Regulatory Short-Termism

Our goal here is to compare two regulatory environments, and related unconstrained allocation strategies (a) and (b), with (b) being a long-termist environment, and (a) being a more or less short-termist environment. In the environment (a), the pension fund invests A_0 in the unconstrained strategy and the funding ratio is checked after δt years. If the minimum funding constraint is not satisfied at this date, checking takes place every year until the funding ratio is back above the limit. As soon as a sufficiently funded status is recovered, checking takes place every δt years, and so on. Moreover, if at some checking date the constraint is not met, the sponsor is called to a contribution whose level is $1/m$ times the current deficit. A last checking is made at terminal date T: if the constraint is not satisfied, the sponsor has to fill the gap. In environment (b), the pension fund invests $A_0 + x$ in the unconstrained strategy, the funding ratio is checked only at terminal date T (and the sponsor has to fill the gap immediately if necessary, according to our assumption). The total cost of strategy (a) is $A_0 + C_0(\delta t, m, A_0)$, and the total cost of (b) is $A_0 + x + C_0(1, T, A_0 + x)$. We compute the value of x that makes expected utilities (with respect to utility function u) from terminal funding ratios in (a) and (b) equal. We denote this particular value with $a_{\mathrm{eq}}(\delta t, m, A_0)$. We then compare the cost of (a) with the cost of (b) when $x = a_{\mathrm{eq}}(\delta t, m, A_0)$ by measuring the welfare gain/loss of the short-termist regulatory environment versus the looser regulatory environment through the following quantity (relative to initial asset value):

$$\Delta = a_{\mathrm{eq}}(\delta t, m, A_0) + C_0(T, 1, A_0 + a_{\mathrm{eq}}(\delta t, m, A_0)) - C_0(\delta t, m, A_0)$$

Table 4 about here

Results are reported in table 4, where they are all expressed as a percentage of initial asset value. We find that the present value of future contribution decreases when δt and m increase, which was to be expected and confirms that short-termism is costly since it induces more contributions. On the other hand, one should also recognize that early contributions might lead to higher terminal funding ratios. For example, the very tight regulatory environment $\delta t = 1$ (funding ratios assessed every year) and $m = 1$ (immediate funding of the deficit) is costly (for $\gamma = 2$, the cost of additional contributions is 28.18%, to be compared to 15.71%

when $\delta t = 10$), but leads to a better (in terms of expected utility) funding ratio compared to the loose regulation that consists in measuring the funding ratio only at horizon ($\delta t = T$), as can be seen from the fact that the certainty equivalent amount $a_{\text{eq}}(1, 1, A_0)$ is positive, and relatively high (11.18%). More generally, we find that $a_{\text{eq}}(\delta, m, A_0)$ decreases when δt and m increase, which also is intuitive. Beside, $C_0(T, 1, A_0 + a_{\text{eq}}(\delta t, m, A_0))$ increases when $a_{\text{eq}}(\delta t, m, A_0)$ decreases, which is natural since the initial wealth is lower, and therefore there are more states of the world that require a contribution at the final date. We also see that Δ is a decreasing function of δt and m. This has a very natural interpretation: the cost of short-termism decreases when regulation is less short-termist, whether it controls funding status less frequently or it leaves more time for refunding of deficits. Overall, the impact in terms of Δ shows that short-termism is costly but that the induced cost is limited. Focusing on the case $\gamma = 2$, we find that in the worst case, that is when the regulation is highly short-termist ($\delta t = 1$ and $m = 1$), the welfare loss is limited to -4.607%. Realistic regulatory environments, where funding ratios are measured every year and recovery times are at the minimum equal to 3 years, are in fact more accurately captured by $\delta t = 1$ and $m = 3$, with a cost of short-termism that goes from -1.623% for $\gamma = 2$ to -0.092% for $\gamma = 10$. Overall, absolute values of Δ are relatively small, albeit economically significant. These results provide some interesting perspective on the current debate regarding pension fund regulation, as they suggest that a short-termist regulation is costly in terms of welfare for the pension fund but that the welfare loss is relatively small.

4 Optimal Allocation Decisions in the Presence of Funding Ratio Constraints

As discussed before, funding ratio constraints, whether desirable or not, are dominant in pension funds' environment. The allocation strategy presented in section 2 is in fact not optimal in the presence of minimum funding requirements.

4.1 Portfolio Optimization with Funding Ratio Constraints

As opposed to following an unconstrained strategy, and eventually requesting additional contributions, we now turn to the analysis of the optimal allocation strategy when funding ratio constraints are explicitly accounted for. To this end, we now consider the following optimization program with explicit constraints:

$$\max_{\boldsymbol{\omega}} \mathbb{E}\left[u(F_T)\right], \quad \text{s.t. } A_T \geq kL_T \tag{4.1}$$

Imposing such an explicit lower bound intuitively means that the pension fund has infinitely low utility from funding ratios below k. In fact it can be shown that (4.1) is equivalent to the following program (see Bouchard et al. (2004) for portfolio selection problems with non-smooth preferences):

$$\max_{\boldsymbol{\omega}} \mathbb{E}\left[\widetilde{u}^k(F_T)\right] \tag{4.2}$$

where $\widetilde{u}^k$ is a non-smooth utility function defined as follows:

$$\widetilde{u}^k(x) = \begin{cases} u(x) & \text{if } x \geq k \\ -\infty & \text{if } x < k \end{cases} \tag{4.3}$$

It should be noted that because of the budget constraint (2.5), we need to have $A_0 - L_0^{0,T} \geq kL_0^{T,T_0}$ in order to get $F_T \geq k$ almost surely. Note also that the complete market assumption is critical here, since the presence of a non hedgeable source of risk would make it impossible for the terminal constraint to hold almost surely.

The following proposition provides the solution to the optimization process in the presence of funding ratio constraints.

Proposition 3 • *In the generic case, the optimal payoff in* (4.1) *is:*

$$A_T^{*k} = kL_T + (\xi A_T^{*u} - kL_T)^+$$

• *In the zero-coupon case, the optimal strategy is given by:*

$$\begin{aligned}\boldsymbol{\omega}_t^{*k} = {} & \left[\alpha(T_0 - t)\boldsymbol{e}'_{n+2}\boldsymbol{\sigma}_t^{-1}\boldsymbol{\sigma}_r + \boldsymbol{e}'_{n+2}\boldsymbol{\sigma}_t^{-1}\boldsymbol{\sigma}_\Phi\right]\left[1 - \frac{1}{\gamma}\left(1 - \frac{k\mathscr{N}(-d_{2,t})L_t}{A_t^{*k}}\right)\right]\boldsymbol{\omega}_t^{LMP}(T, T_0) \\ & + \frac{\boldsymbol{e}'_{n+2}\boldsymbol{\sigma}_t^{-1}\boldsymbol{\lambda}}{\gamma}\left(1 - \frac{k\mathscr{N}(-d_{2,t})L_t}{A_t^{*k}}\right)\boldsymbol{\omega}_t^{PSP}\end{aligned}$$

where:

$$d_{1,t} = \frac{1}{\frac{1}{\gamma}\sqrt{\int_t^T \|\boldsymbol{\sigma}_I(s,T_0) - \boldsymbol{\lambda}\|^2\, ds}}\left[\ln\frac{\xi A_t^{*u}}{kL_t} + \frac{1}{2\gamma^2}\int_t^T \|\boldsymbol{\sigma}_I(s,T_0) - \boldsymbol{\lambda}\|^2\, ds\right]$$

$$d_{2,t} = d_{1,t} - \frac{1}{\gamma}\sqrt{\int_t^T \|\boldsymbol{\sigma}_I(s,T_0) - \boldsymbol{\lambda}\|^2\, ds}$$

$$A_t^{*k} = kL_t + \mathscr{N}(d_{1,t})\xi A_t^{*u} - kL_t\mathscr{N}(d_{2,t})$$

The constant ξ is chosen so that the budget constraint $A_0^* = A_0$ holds.

Proof. See appendix A.1. ■

We find that the optimal terminal wealth is given by an initial long position kL_0^{T,T_0} in a portfolio with payoff kL_T, while the remainder $A_0 - L_0^{0,T} - kL_0^{T,T_0}$ is invested in an exchange option, which allows the investor to exchange, when the option expires in the money, a portfolio of terminal value kL_T for a portfolio delivering the payoff ξA_T^{*u}, where the constant ξ is adjusted to make the price of the option equal to $A_0 - L_0^{0,T} - kL_0^{T,T_0}$.[16] As a result, the terminal net wealth will be the maximum of kL_T and ξA_T^{*u}. The expression for the optimal investment strategy and wealth process is somewhat reminiscent of OBPI (Option-Based Portfolio Insurance) strategies, which the present setup extends in many dimensions. First, the underlying asset is not a risky asset but the underlying optimal unconstrained strategy. Also, the risk-free asset is not cash but the liability-benchmark, and exchange option flavor, which allows one to transport the structure to an asset-liability relative risk management context.

A comparison between the optimal terminal wealth under the unconstrained strategy and the constrained strategy can be found in the following proposition, which formalizes the intuition according to which insurance of downside risk (relative to liabilities) has a cost in terms of performance potential.

Proposition 4 *For the states of the world* ω *such that* $F_T^{*k}(\omega) \equiv \frac{A_T^{*k}}{L_T}(\omega) > k$*, or equivalently such that* $[\xi A_T^{*u}(\omega) - kL_T(\omega)]^+ > 0$*, we have that* $A_T^{*k}(\omega) < A_T^{*u}(\omega)$*.*

Proof. To see this, first note that for $k > 0$, the price of the exchange option lies between the no-arbitrage bounds given on the one hand by the intrinsic value of the option $\left[\xi(A_0 - L_0^{0,T}) - kL_0^{T,T_0}\right]^+$ and on the other hand by the underlying price $\xi(A_0 - L_0^{0,T})$:

$$\left[\xi(A_0 - L_0^{0,T}) - kL_0^{T,T_0}\right]^+ < A_0 - L_0^{0,T} - kL_0^{T,T_0} < \xi(A_0 - L_0^{0,T})$$

which implies that $\xi < 1$ and also that $\xi > 1 - \frac{kL_0^{T,T_0}}{A_0 - L_0^{0,T}}$. ■

4.2 Empirical Analysis

In this subsection 4.2, we return to the case where several liability payments take place before T. We take $T_0 = 75$ years.

Table 5 about here

In table 5, we test for the introduction of explicit funding ratio constraints, with a minimum set at $k = 90\%$. In this case, we find that the minimum funding ratio is indeed limited to k, a

[16]Note, of course, that when k is zero, the constrained optimal terminal wealth A_T^{*k} coincides with the unconstrained optimal terminal net wealth A_T^{*u}.

value that is reached with a relatively high probability, suggesting that the margin for error is fully utilized with these strategies. In fact, the dispersion of the funding ratio distribution is narrower on both sides when the strategy with explicit constraints is implemented compared to the unconstrained counterpart. Hence, for example in the case $T = 20$ years and $\gamma = 5$, the maximum funding ratio is 3.1 in the explicit constraint case while it reaches 4.01 in the unconstrained case. That downside protection comes at the cost of a more limited access to the upside potential can also be seen from the fact that the expected terminal funding ratio conditional upon being larger than the constraint k, $\mathbb{E}\left[F_T^{*k} | F_T^{*k} \geq k\right]$, is always strictly lower in the constrained case compared to the unconstrained case. For example, when $T = 20$ years and $\gamma = 5$, the conditional mean reaches 1.16 in the (explicitly) constrained case, while it is 1.51 in the unconstrained case. On the other hand, the average deficit is also significantly higher in the latter case (13%, as opposed to 8% in the constrained case).

5 Introducing Maximum Funding Ratio Constraints

Given the complexity of surplus sharing rules, it is unclear whether pension funds have any utility over exceedingly large surpluses (see Pugh (2003)). It should also be noted that in some regulatory environments, maximum funding constraints are imposed by tax authorities to prevent the deliberate or accidental build-up of excessive assets within the pension fund. More generally, maximum funding ratios, when they are not a constraint, can be a target. In the Netherlands for example, there exists a target funding ratio that is a function of investment risk, and reaches 130% on average. In this context, and given that pension funds have preferences that likely reach satiation beyond a given funding ratio level, it seems reasonable to try and analyze how the introduction of maximum funding ratio targets would impact the optimal strategy. The idea is that by giving up part of the upside potential beyond levels where marginal utility of wealth (relative to liabilities) is low or almost zero, the investor can decrease the cost of downside protection.

5.1 Portfolio Optimization with Lower and Upper Bounds

We thus consider the simultaneous introduction of a minimum and a maximum funding ratio constraints, so that the optimization program is given by (2.8) subject to the additional constraints $F_T \geq k$ and $F_T \leq k'$:

$$\max_{\boldsymbol{\omega}} \mathbb{E}[u(F_T)] \quad \text{s.t. } k \leq F_T \leq k' \tag{5.1}$$

In order to have the constraint $k \leq F_T \leq k'$ satisfied almost surely, the initial asset A_0

should lie between $L_0^{0,T} + kL_0^{T,T_0}$ and $L_0^{0,T} + k'L_0^{T,T_0}$. A related program reads:[17]

$$\max_{\boldsymbol{\omega}} \mathbb{E}\left[\widetilde{u}^{k,k'}(F_T)\right] \tag{5.2}$$

where $\widetilde{u}$ is a non-smooth utility function defined as follows:

$$\widetilde{u}^{k,k'}(x) = \begin{cases} u(x) & \text{for } k \leq x \leq k' \\ -\infty & \text{for } x < k \\ u(k') & \text{for } x > k' \end{cases} \tag{5.3}$$

The following proposition presents the solution to these optimization programs in the presence of minimum and maximum funding ratio constraints (or targets).

Proposition 5 • *In the generic case, the optimal payoff in* (5.1) *is:*

$$A_T^{*k,k'} = kL_T + (\xi' A_T^{*u} - kL_T)^+ - (\xi' A_T^{*u} - k'L_T)^+$$

• *In the zero-coupon case, the optimal strategy is given by:*

$$\begin{aligned}
\boldsymbol{\omega}_t^{*k,k'} &= \alpha(T_0 - t)\left[1 - \frac{1}{\gamma}\left(1 - \left[k\mathcal{N}(-d_{2,t}) + k'\mathcal{N}(d'_{2,t})\right]\frac{L_t}{A_t^{*k,k'}}\right)\right]\boldsymbol{\sigma}_t^{-1}\boldsymbol{\sigma}_r \\
&\quad + \left[1 - \frac{1}{\gamma}\left(1 - \left[k\mathcal{N}(-d_{2,t}) + k'\mathcal{N}(d'_{2,t})\right]\frac{L_t}{A_t^{*k,k'}}\right)\right]\boldsymbol{\sigma}_t^{-1}\boldsymbol{\sigma}_\Phi \\
&\quad + \frac{1}{\gamma}\left(1 - \left[k\mathcal{N}(-d_{2,t}) + k'\mathcal{N}(d'_{2,t})\right]\frac{L_t}{A_t^{*k,k'}}\right)\boldsymbol{\sigma}_t^{-1}\boldsymbol{\lambda}
\end{aligned}$$

where:

$$\begin{aligned}
d'_{1,t} &= \frac{1}{\frac{1}{\gamma}\sqrt{\int_t^T \|\boldsymbol{\sigma}_I(s,T_0) - \boldsymbol{\lambda}\|^2\, ds}}\left[\ln\frac{\xi' A_t^{*u}}{k'L_t} + \frac{1}{2\gamma^2}\int_t^T \|\boldsymbol{\sigma}_I(s,T_0) - \boldsymbol{\lambda}\|^2\, ds\right] \\
d'_{2,t} &= d'_{1,t} - \frac{1}{\gamma}\sqrt{\int_t^T \|\boldsymbol{\sigma}_I(s,T_0) - \boldsymbol{\lambda}\|^2\, ds} \\
A_t^{*k,k'} &= kL_t\mathcal{N}(-d_{2,t}) + k'L_t\mathcal{N}(d'_{2,t}) + [\mathcal{N}(d_{1,t}) - \mathcal{N}(d'_{1,t})]\xi' A_t^{*u}
\end{aligned}$$

The constant ξ' is adjusted to make the budget constraint $A_0^{*k,k'} = A_0$ hold.

[17] The existence theorems in Schachermayer (2001) or in Bouchard et al. (2004) suggest the solution to (5.2) is identical to the solution to (5.1). In other words, in program (5.2), one optimally implements a strategy that will not lead to funding ratios above k', even though this is not strictly forbidden as in program (5.1). Intuitively, this is because asset allocation decisions leading to funding ratios beyond k' involve an additional risk while they do not involve any marginal utility gain.

Proof. See appendix A.2. ∎

The optimal terminal net wealth is given by the payoff of a static portfolio strategy that consists of investing kL_0 in the liability-matching portfolio, with the remainder, $A_0 - kL_0$, being invested in a *bull spread* strategy extended to an exchange option context. This strategy consists of a long position in an exchange option and a short position in an exchange option on the same underlying payoff and a higher strike price. The former option gives its owner the right to exchange a portfolio of terminal value kL_T for a portfolio of terminal value $\xi' A_T^{*u}$, while the latter option gives its owner the right to exchange a portfolio of terminal value $k' L_T$ for the same portfolio of terminal value $\xi' A_T^{*u}$.

Comparing propositions 3 and 5, it can be seen that the imposition of an explicit upper bound involves a short position in an exchange option. This short position allows one to reduce the cost of downside protection by giving up some access to the upside potential beyond the funding ratio threshold k', as shown in the following proposition.

Proposition 6 *Let $F_T^{*k,k'} \equiv \frac{A_T^{*k,k'}}{L_T}$ denote the optimal terminal funding ratio when the lower bound k and the upper bound k' are imposed. Similarly, let $F_T^{*k} \equiv \frac{A_T^{*k}}{L_T}$ denote the optimal terminal funding ratio when only the lower bound k is imposed. For those states of the world ω such that $k < F_T^{*k,k'}(\omega) < k'$, we have that $A_T^{*k}(\omega) < A_T^{*k,k'}(\omega)$.*

Proof. By definition of ξ', we have that:

$$
\begin{aligned}
&Ex(0, r_0, \Phi_0, \xi(A_0 - L_0^{0,T}), kL_0^{T,T_0}) = \\
&\qquad Ex(0, r_0, \Phi_0, \xi'(A_0 - L_0^{0,T}), kL_0^{T,T_0}) - Ex(0, r_0, \Phi_0, \xi'(A_0 - L_0^{0,T}), k'L_0^{T,T_0})
\end{aligned}
$$

Since Ex is a strictly increasing function of its fourth argument, we obtain that $\xi(A_0 - L_0^{T,T_0}) < \xi'(A_0 - L_0^{T,T_0})$, which implies that $\xi < \xi'$. The proposition follows from this inequality. ∎

5.2 Empirical Analysis

In this subsection 5.2, several liability payments take place, and we take $T_0 = 75$ years.

Table 6 about here

In table 6, we introduce an additional upper bound constraint, with a maximum funding ratio value set at $k' = 110\%$. Giving up access to the upside potential above 110% allows one to decrease the cost of downside protection, as can be seen from the fact that the average of terminal funding ratio values conditional upon being in the range between 90% and 110% are higher when the upper bound is introduced compared to when it is not introduced. In fact, focusing again on $T = 20$ years and $\gamma = 5$, we have that the conditional expected funding ratio $\mathbb{E}\left[F_T^{*k,k'}|k \leq F_T^{*k,k'} \leq k'\right] = 1.04$ when both constraints are imposed, while it merely reached 0.95 when only the lower constraint was imposed. Comparing the solution with both constraints to the unconstrained case from table 3, we find in the unconstrained case that $\mathbb{E}\left[F_T^{*k,k'}|k \leq F_T^{*k,k'} \leq k'\right] = 1.01 < 1.04$. Hence, the addition of the short option position allows for an increase in the mean funding ratio on the range of values between 0.9 and 1.1, not only with respect to the case with minimum funding requirement only, but also with respect to the unconstrained case.

6 The (Ir)relevancy of Risk Management

In practice, pension funds typically do not implement the risk-controlled strategies introduced in section 4, even though these were shown to be optimal in the presence of funding ratio constraints. The alternative to these risk-controlled strategies consists of following unconstrained allocation strategies, and requiring additional contributions from the sponsor company when and if needed for bringing the funding ratio back to the minimum required level.[18] A formal analysis of the welfare cost induced by this alternative is of high practical interest and relevance, and is subject of this section.

6.1 An Irrelevancy Principle

In what follows, we are able to isolate a number of conditions under which *risk-management is irrelevant*, in the sense that both approaches lead to the exact same terminal payoffs. This irrelevancy principle is, however, subject to a number of conditions, and we obtain as a corollary that any deviation from these conditions will induce a utility cost, that we will assess in subsection 6.2.

6.1.1 The Case of Minimum Funding Ratio Constraints

If the fund implements an unconstrained strategy and is subject to a minimum funding constraint of k at date T, the final contribution for the sponsor company is $\text{Cont}_T = (kL_T - A_T^{*u,A_0})^+$. (Note that in this expression, and in the analysis that follows, we make explicit the dependence upon A_0 in the notation for the optimal asset value process A_T^{*u,A_0} that was introduced in Proposition 2.) The cost of downside protection for the sponsor company can therefore be measured as the initial price, denoted by C_0, of an option with payoff

[18]Note that nothing guarantees that the unconstrained strategy, which is optimal in the absence of funding ratio constraints, remains optimal when the presence of additional contributions from the sponsor company are accounted for.

$\text{Cont}_T = (kL_T - A_T^{*u,A_0})^+$. In other words, the cost of the strategy that consists of investing in the unconstrained strategy and holding on the side an option with payoff $(kL_T - A_T^{*u,A_0})^+$ is $A_0 + C_0$, and the payoff of this strategy is $\max(A_T^{*u,A_0}, kL_T)$. We would like to compare this payoff to that obtained through a strategy enforcing downside risk protection. The following proposition presents an *irrelevance principle*, that shows that the two strategies are in fact strictly equivalent.

Proposition 7 *Let*

$$C_0 = \mathbb{E}^{\mathbb{Q}}\left[\exp\left(-\int_0^T r_s\, ds\right)\left(kL_T - A_T^{*u,A_0}\right)^+\right]$$

denote the initial cost of the guarantee.

Then the following two strategies lead to the same payoff: a) require a guarantee with payoff $(kL_T - A_T^{*u,A_0})^+$ *from the sponsor company, with the initial cost* C_0*, and invest* A_0 *in an optimal unconstrained allocation strategy; b) require no guarantee and invest* $A_0 + C_0$ *in an optimal allocation strategy subject to explicit minimum funding constraint* $F_T \geq k$*.*

Proof. We take the present value of the equality between $\left(kL_T - A_T^{*u,A_0}\right)^+$ and $kL_T - A_T^{*u,A_0} + \left(A_T^{*u,A_0} - kL_T\right)^+$, which leads to the following relation, which is reminiscent of the put-call parity equality, transposed to the context of exchange options:

$$C_0 = kL_0^{T,T_0} - A_0 + L_0^{0,T} + Ex(0, r_0, \Phi_0, A_0 - L_0^{0,T}, kL_0^{T,T_0})$$

But we also have, by definition of ξ:

$$A_0 + C_0 - L_0^{0,T} = kL_0^{T,T_0} + Ex(0, r_0, \Phi_0, \xi(A_0 + C_0 - L_0^{0,T}), kL_0^{T,T_0})$$

We thus get that $\xi(A_0 + C_0 - L_0^{0,T}) = A_0 - L_0^{0,T}$. From the expression of the optimal terminal unconstrained asset value (see proposition 2), this yields $\xi A_T^{*u,A_0+C_0} = A_T^{*u,A_0}$, which concludes the proof. ■

This result seems to suggest that risk management is irrelevant; as opposed to implement risk-controlled dynamic allocation strategies, the pension fund can simply follow an unconstrained strategy, and eventually require an additional contribution if and when needed. A number of comments should, however, made about this conclusion. First it should be noted that, in a complete market environment, the approach aiming at enforcing downside protection through dynamic allocation strategies allows for a respect of the minimum funding ratio requirement not only at terminal date, but also at all possible dates. This stands in sharp contrast to the approach aiming at requiring a final contribution, which can involve large deficits at intermediate dates. The key difference between enforcing minimum funding ratio constraints at the terminal date versus imposing them at all dates is that in the latter situation, the pension fund can not expect that good performance on the risky asset classes will help resorb the deficit. The difference will be interpreted as the cost of short-termism. Rather surprisingly, and in contrast to a widespread belief, we find that short-termism induces only a very mild cost.

Furthermore, it should be noted, as an additional advantage of allocation versus contribution strategies, that the efficiency of not managing risk, and requiring additional contributions, is seriously impaired by the fact that contributions are irreversible. In other words, if the pension fund ends up with very large surplus, it will typically prove very complex for the sponsor company to enjoy a "negative contribution", that is to withdraw money from the pension fund. In practice, the best it can hope for is a contribution holiday. On the other hand, dynamic allocation strategies can be designed to achieve a payoff that will never exceed a given fraction of the liability value. We investigate this point in the next subsection.

6.1.2 The Case of Minimum and Maximum Funding Ratio Constraints

In the spirit of subsection 6.1.1, we wish to compare the payoff obtained by imposing the double constraint to the payoff achieved through an unconstrained strategy coupled with a state-dependent contribution from the sponsor at the terminal date. We show in the following proposition that the dynamic asset allocation strategy with double constraints would be formally equivalent to the unconstrained strategy with reversible contribution at the terminal date, if such a negative contribution could take place.

Proposition 8 *Let*

$$C_0' = \mathbb{E}^{\mathbb{Q}}\left[\exp\left(-\int_0^T r_s\,ds\right)\left(kL_T - A_T^{*u,A_0}\right)^+\right] - \mathbb{E}^{\mathbb{Q}}\left[\exp\left(-\int_0^T r_s\,ds\right)\left(A_T^{*u,A_0} - k'L_T\right)^+\right]$$

be the initial cost of the contribution. Then the following two strategies lead to the same payoff: a) require a (possibly negative) contribution with payoff $(kL_T - A_T^{*u,A_0})^+ - (A_T^{*u,A_0} - k'L_T)^+$ from the sponsor company, with the initial cost C_0', and invest A_0 in an optimal unconstrained allocation strategy; b) require no contribution and invest $A_0 + C_0'$ in an optimal allocation strategy subject to explicit minimum and maximum funding constraints $k \leq F_T \leq k'$.

Proof. When the sponsor contributes at date T, the final net asset is $A_T^1 = A_T^{*u,A_0} + \mathrm{Cont}_T'$ where the final reversible contribution is $\mathrm{Cont}_T' = (kL_T - A_T^{*u,A_0})^+ - (A_T^{*u,A_0} - k'L_T)^+$. The present value of this contribution is:

$$C_0' = kL_0^{T,T_0} - A_0 + L_0^{0,T} + Ex(0, r_0, \Phi_0, A_0 - L_0^{0,T}, kL_0^{T,T_0}) - Ex(0, r_0, \Phi_0, A_0 - L_0^{0,T}, k'L_0^{T,T_0})$$

Consider now the situation where the pension fund starts from initial asset $A_0 + C'_0$ and follows the optimal strategy subject to the double constraint. The final asset value is then $A^2_T = kL_T + (\xi' A_T^{*u,A_0+C'_0} - kL_T)^+ - (\xi' A_T^{*u,A_0+C'_0} - k'L_T)^+$ where the constant ξ' must satisfy:

$$\begin{aligned} A_0 + C'_0 - kL_0^{T,T_0} - L_0^{0,T} = Ex(0, r_0, \Phi_0, \xi'(A_0 + C'_0 - L_0^{0,T}), kL_0^{T,T_0}) \\ - Ex(0, r_0, \Phi_0, \xi'(A_0 + C'_0 - L_0^{0,T}), k'L_0^{T,T_0}) \end{aligned}$$

We thus have that:

$$\begin{aligned} Ex(0, r_0, \Phi_0, \xi'(A_0 + C'_0 - L_0^{0,T}), kL_0^{T,T_0}) - Ex(0, r_0, \Phi_0, \xi'(A_0 + C'_0 - L_0^{0,T}), k'L_0^{T,T_0}) \\ = Ex(0, r_0, \Phi_0, A_0 - L_0^{0,T}, kL_0^{T,T_0}) - Ex(0, r_0, \Phi_0, A_0 - L_0^{0,T}, k'L_0^{T,T_0}) \quad (6.1) \end{aligned}$$

Matching the initial values of these two bull-spread strategies yields $\xi'(A_0 + C'_0 - L_0^{0,T}) = A_0 - L_0^{0,T}$, which implies that $\xi' A_T^{*u,A_0+C'_0} = A_T^{*u,A_0}$ almost surely. We thus obtain that $A^1_T = A^2_T$ $\mathbb{P}$-a.s., which completes the proof. ■

One may define the *cost of irreversibility* as the difference between C_0, which is the present value of a single contribution at time T when irreversibility prevents the sponsor from recovering the assets in excess of kL_T, and C'_0, which is the present value of a single contribution at time T when reversibility allows the sponsor to recover the excess of assets over minimum threshold. In other words, we set C'_0 such that:

$$C_0 - C'_0 = \mathbb{E}^{\mathbb{Q}}\left[\exp\left(-\int_0^T r_s\,\mathrm{d}s\right)\left(A_T^{*u,A_0} - k'L_T\right)^+\right]$$

As outlined in the introduction, a second reason why implementing risk-management through dynamic allocation strategies differs from implementing risk-management through additional contribution strategies is related to the presence of *short-term* minimum funding requirements, which impose that additional contributions occur every year if the minimum funding ratio is not met, thus preventing the sponsor company to enjoy the possibility for the fund to get back to solvency levels following good financial market performance. In the next section, we perform a separate numerical assessment of the impact of contribution irreversibility and short-termism when comparing unconstrained strategies to constrained strategies.

6.2 Measuring the Costs and Benefits of Risk Management

We now wish to compare the following two strategies: (a) the pension fund invests A_0 in the unconstrained strategy of Proposition 2, the funding ratio is checked all δt years as long as the funding ratio stays above the limit, and every year while it lies below, and the sponsor has to finance $1/m$ of the current deficit each year; (c) the pension fund invests $A_0 + x$ in the *constrained* strategy and the sponsor does not (need to) contribute. The constrained strategy in (c) can be either with a minimum funding ratio constraint only or with both minimum and maximum funding ratio constraints. As before, we compute the value, which we denote by $a^c_{\text{eq}}(\delta t, m, A_0)$, of x that makes expected utilities from terminal funding ratios in (a) and (c) equal. We then compare the cost of (a) with the cost of (c) when $x = a^c_{\text{eq}}(\delta t, m, A_0)$ by measuring the welfare loss of non-implementing a risk-controlled strategy as:

$$\Delta_c = a^c_{\text{eq}}(\delta t, m, A_0) - C_0(\delta t, m, A_0)$$

It can be shown that if the pension fund has utility function $\widetilde{u}^k$, the expected utility from the terminal funding ratio is a strictly increasing function of A_0.[19] A consequence is that when $\delta t = T$, $a^c_{\text{eq}}(\delta t, m, A_0)$ is unique if it exists. Moreover, Proposition 7 implies that in the case of a single lower constraint, $a^c_{\text{eq}}(T, 1, A_0)$ is equal to $C_0(T, 1, A_0)$. Note that the costs of additional contributions, denoted by C^c_0, are always 0 for the constrained strategies since they precisely have been designed to avoid with probability 1 funding ratios below the lower limit k.

In fact, it can be argued that following a risk-controlled strategy precisely amounts to turn short-term constraints into long-term constraints (i.e., transform $\delta t < T$ into $\delta t = T$). As a result, the cost of non following a risk-controlled strategy is identical to the cost of short-termism, i.e., $\Delta_c = \Delta$, and has therefore already been estimated in table 4. To see this more formally, first note that (again as a consequence of proposition 7) investing $A_0 + a_{\text{eq}}(\delta t, m, A_0) + C_0(T, 1, A_0 + a_{\text{eq}}(\delta t, m, A_0))$ in the constrained strategy yields the same payoff, hence the same expected utility, as investing $A_0 + a_{\text{eq}}(\delta t, m, A_0)$ in an unconstrained strategy and requiring a unique final contribution from the sponsor at time T. Then, by definition of $a_{\text{eq}}(\delta t, m, A_0)$, investing $A_0 + a_{\text{eq}}(\delta t, m, A_0) + C_0(T, 1, A_0 + a_{\text{eq}}(\delta t, m, A_0))$ in the constrained strategy with a funding ratio checked only at date T leads to the same expected utility as investing A_0 in an unconstrained strategy with funding ratio checked every δt and a recovery time of m years (and subsequently adding contributions with present value $C_0(\delta t, m, A_0)$). Hence, the value $a^c_{\text{eq}}(\delta t, m, A_0)$ that makes expected utilities from (a) and (c) equal is simply the sum of $a_{\text{eq}}(\delta t, m, A_0)$ and $C_0(T, 1, A_0 + a_{\text{eq}}(\delta t, m, A_0))$, two quantities that are displayed in table 5, and we have that $\Delta_c = \Delta$. In table 7, we thus provide estimates for $a^c_{\text{eq}}(\delta t, m, A_0)$ and $C_0(\delta t, m, A_0)$, expressed as percentage of the initial asset value A_0, for different values of γ, δt and m. Hence, we can conclude that the cost of non following a risk-controlled strategy, which is identical to the cost of short-termism, is relatively small but economically significant, for example equal to -4.607% of the initial asset value for $\gamma = 2$ and $\delta t = m = 1$.

[19]This result is intuitive since the terminal funding ratio is itself a strictly increasing pathwise function of A_0, but it deserves a formal proof since u^k is not strictly monotonous. This proof can be obtained from the authors upon request.

Table 7 about here

As before, we now implement the same exercise, but we consider the imposition of a double explicit constraint rather than a single one, thus recognizing that there is little marginal utility from reaching very high funding ratios. Since imposing a double constraint amounts to assuming that the pension fund has utility function $\widetilde{u}^{k,k'}$, expected utilities from both strategies (a) and (c) are computed using this function. Again, we compute the amount $a^c_{\text{eq}}(\delta t, m, A_0)$ that needs to be added to A_0 in order to get the same expected utility from strategy (a) as from strategy (c). In table 8, we provide information about our measure Δ_c of welfare gains/losses induced by implementing, versus not implementing, risk management strategies. These results are obtained for different values of the risk-aversion parameter, when the pension fund horizon is assumed to be $T = 10$ years, $k = 90\%$, and k' is taken to be equal to be 100%, 110% or 130%. The values we obtain now provide a very different picture, and show a very significant cost associated with the absence of risk management, a cost that reaches -30.11% (as a percentage of the initial asset value) when $\gamma = 2$, $\delta t = 1$, $m = 1$ and $k' = 1.1$. This suggests that risk-controlled strategies with minimum and maximum constraints have the appealing property to allow for significant upside benefits at a relatively low cost. From an intuitive standpoint, these results can be explained by the fact that the cost of contribution irreversibility is now added to the cost of regulatory short-termism. In fact, Proposition 8 shows that by following the optimal constrained dynamic strategy, the pension fund allows the sponsor company to save an amount which is exactly equal to the present value of negative contributions that would be enjoyed by the sponsor company if it could withdraw the excess value of assets with respect to a target funding level. In other words, the welfare gains associated with risk management are large because the risk-controlled strategies now not only avoid additional contributions by respecting the minimum funding requirement, but also avoid unnecessarily high funding ratios to which is associated no utility improvement.

Table 8 about here

Overall, and taken together, the results we obtain strongly suggest that risk-controlled strategies can add significant value when compared to unconstrained strategies that result in additional short-term and irreversible contributions from the sponsor company. In the end, it appears that what is costly is not so much the presence of funding ratio constraints per se, but rather the reluctance to follow the dynamic risk-controlled allocation strategies that are optimal given these regulatory constraints.

7 Conclusion and Extensions

Defined-benefit pension funds are currently facing a serious challenge and dilemma. On the one hand, the desire to alleviate the burden of contributions leads them to invest significantly in equity markets and other classes poorly correlated with liabilities but offering superior long-term performance potential. On the other hand, stricter regulatory environments and accounting standards give them more incentive to invest a dominant fraction of their portfolios in assets highly correlated to liabilities. While there is a general agreement about the fact that *some* regulatory constraints are needed, there is a fierce debate regarding whether it makes sense to impose *short-term* constraints to long-term investors. The introduction of funding ratio constraints has been particularly criticized by a number of experts, who find that imposing such short-term constraints to long-term investors could be counter-productive. The point of our paper is not to question whether minimum funding ratio constraints are desirable or not, but instead to try and provide a formal measure of the cost of regulatory short-termism. We analyze this question in the context of a continuous-time dynamic asset allocation model for an investor facing liability commitments subject to inflation and interest rate risks. Perhaps surprisingly, we first find that short-term funding ratio constraints do not involve significant welfare losses, especially when marginal utility decreases sharply beyond a given threshold. Recognizing that the presence of minimum funding ratio constraints, whether desirable or not, should affect the optimal allocation policy, we provide the formal solution to the asset allocation problem in the presence of such constraints. We then compare these risk-controlled strategies to unconstrained allocation strategies coupled with additional contributions, and find that the latter involve severe welfare costs in the presence of irreversible contributions and regulatory short-termism. Overall, our results suggest that it is not so much the presence of short-term funding ratio constraints that is costly per se for pension funds, but rather their reluctance to implement risk-management strategies that are optimal given such regulatory constraints. In essence, we results show that risk-management strategies can turn reversible contributions and short-term constraints into irreversible contributions and long-term constraints, hence the severe opportunity cost for pension funds that do not follow them.

This analysis suffers from a number of caveats. First it should be emphasized that our results, and in particular the ability to turn short-term into long-term constraints, relies on the assumption of market completeness: it is only with complete markets that a risk-controlled strategy that allows for the respect of a minimum funding ratio requirement at the horizon also allows for the respect of the minimum funding ratio constraint at each possible date. To test for the impact of market incompleteness, we have performed robustness checks with respect to the introduction of various forms of market incompleteness (the results of which are omitted for the sake of brevity but can be obtained from the authors upon request), and have found that the risk-controlled strategies perform relatively well in an incomplete market setting with either the introduction of jump risk on the asset side, or the introduction of actuarial risk on the liability side. Also, we have focused on the pension fund situation, mostly taken in isolation from the sponsor company. In fact, we have mostly focus on the perspective of the pension fund manager, who has been assumed to have preferences over terminal funding ratio of the pension fund. We have also avoided to model the preferences of the shareholder of the sponsor company by introducing an implicit disutility over additional contributions, as can be seen from the fact that our measure of the cost of regulatory short-termism (respectively, of the reluctance to follow risk-controlled strategies) was taken to be the present value of additional contributions needed to achieve a level of expected utility identical to that obtained with long-term regulatory constraints (respectively, to that obtained with risk-controlled strategies). Abstracting away from the sponsor company allows for a dramatic simplification of the problem, and it would be desirable to develop a more integrated approach to asset-liability management, with a focus on optimal allocation and contribution policies from the shareholders' standpoint. This would however require a formal capital structure model, with an analysis of the rational valuation of liability streams as defaultable claims, as well as an analysis of the impact of asset allocation decisions on the sponsor company credit ratings. It would also require a careful analysis of the agencies issues amongst the various stakeholders, including equity holders and bondholders of the sponsor company, but also workers and pensioners, as well as managers of the pension funds and their trustees. Regarding default on pension obligations, it should be noted that in some countries like the US, there exists a pension insurance system, which is in charge of (partially) compensating for the deficit, if any, in pension payment in case of default from the sponsor company.[20] Analyzing the impact of these additional sources of complexity is beyond the scope of the present paper, and is left for further research.

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[20]See Bodie (1996) for a discussion of the pension put, and also Bodie et al. (1985) for empirical evidence that PBCG creates an incentive for distressed companies to underfund their pension plan and invest in risky assets.

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A Proof of the Main Propositions

A.1 Proof of proposition 3

The first-order optimality condition reads:

$$\frac{1}{L_T}\left(\frac{A_T^{*k}}{L_T}\right)^{-\gamma} - \nu_2 M_T + \frac{\nu_3}{L_T} = 0$$

where $\nu_3 \geq 0$, $\nu_3\left(\frac{A_T^{*k}}{L_T} - k\right) = 0$ and $A_T^{*k} \geq kL_T$. Hence the optimal terminal net wealth and the optimal wealth process:

$$A_T^{*k} = kL_T + \left[L_T\left(\nu_2 M_T L_T\right)^{-\frac{1}{\gamma}} - kL_T\right]^+ \tag{A.1}$$

$$A_t^{*k} = L_t^{t,T} + kL_t^{T,T_0} + \mathbb{E}_t^{\mathbb{Q}}\left[e^{-\int_t^T r_u\,\mathrm{d}u}(\xi A_T^{*u} - kL_T)^+\right] \tag{A.2}$$

where A_T^{*u} is the optimal final net wealth in the absence of constraints on the funding ratio, ξ is equal to $(\nu_2/\nu_0)^{-\frac{1}{\gamma}}$ and ν_0 is the Lagrange multiplier associated to the budget constraint in (2.8).

The expectation in (A.2) is the price at time t of the option to exchange the payoff kL_T for the payoff ξA_T^{*u} at time T. It is only in the zero-coupon case that the volatility vector of A^{*u}/L is deterministic, which allows for the use of Margrabe's formula to price the exchange option:

$$A_t^{*k} = kL_t + \xi(A_t^{*u} - L_t^{t,T})\mathscr{N}(d_{1,t}) - kL_t^{T,T_0}\mathscr{N}(d_{2,t})$$

where $d_{1,t}$ and $d_{2,t}$ are given in the statement of the proposition. We then apply Ito's lemma:

$$\mathrm{d}A_t^{*k} = r_t A_t^{*k}\,\mathrm{d}t + kL_t\boldsymbol{\sigma}_I(t,T_0)'\,\mathrm{d}\boldsymbol{z}_t + (A_t^{*k} - kL_t)\boldsymbol{\sigma}_I(t,T_0)'\,\mathrm{d}\boldsymbol{z}_t + \frac{\xi(A_t^{*u} - L_t^{t,T})\mathscr{N}(d_{1,t})}{\gamma}[\boldsymbol{\lambda} - \boldsymbol{\sigma}_I(t,T_0)]'\,\mathrm{d}\hat{\boldsymbol{z}}_t$$

Writing $\xi A_t^{*u}\mathscr{N}(d_{1,t})$ as $A_t^{*k} - kL_t\mathscr{N}(-d_{2,t})$ and rearranging terms leads to the optimal portfolio strategy given in Proposition 3.

A.2 Proof of proposition 5

We just explain how to derive the optimal terminal net wealth. The first-order optimality condition for the optimization program reads:

$$\frac{1}{L_T}\left(\frac{A_T^{*k,k'}}{L_T}\right)^{-\gamma} - \nu_4 M_T + \frac{\nu_5}{L_T} - \frac{\nu_6}{L_T} = 0$$

where $kL_T \leq A_T^* \leq k'L_T$, ν_5 and ν_6 are nonnegative and $\nu_5(A_T^{*k,k'} - kL_T) = \nu_6(A_T^{*k,k'} - k'L_T) = 0$. This implies that:

$$A_T^{*k,k'} = k + \left[(\nu_4 M_T L_T)^{-\frac{1}{\gamma}} - k\right]^+ - \left[(\nu_4 M_T L_T)^{-\frac{1}{\gamma}} - k'\right]^+$$

Having written the optimal payoff under this form, we are back to the proof of Proposition 3 (see appendix A.1).

Table 1: Schedule of annual liability payments expressed in real terms.

Year	Payment	Year	Payment	Year	Payment	Year	Payment
1	6891.04	21	4620.24	41	1114.46	61	52.1
2	7080.01	22	4422.07	42	1008.22	62	40.86
3	7086.14	23	4233.09	43	908.11	63	32.69
4	7034.05	24	4043.1	44	814.14	64	25.54
5	6980.93	25	3822.45	45	727.31	65	19.41
6	6900.23	26	3598.74	46	646.61	66	15.32
7	6767.44	27	3383.21	47	572.04	67	11.24
8	6704.1	28	3173.8	48	503.6	68	8.17
9	6631.58	29	2976.65	49	440.27	69	6.13
10	6542.7	30	2785.63	50	383.06	70	4.09
11	6435.45	31	2597.67	51	330.97	71	3.06
12	6285.29	32	2413.8	52	283.98	72	2.04
13	6113.68	33	2240.15	53	242.1	73	1.02
14	5940.02	34	2074.67	54	205.32	74	1.02
15	5754.11	35	1914.29	55	172.63	75	1.02
16	5575.34	36	1761.06	56	144.03	76	0
17	5393.52	37	1616.01	57	119.52	77	0
18	5195.35	38	1479.13	58	98.06	78	0
19	5024.76	39	1350.42	59	79.68	79	0
20	4830.67	40	1228.86	60	64.35	80	0

This table presents the schedule of annual pension payments expressed in real terms. The data has been provided for by a Dutch pension fund. The duration of this payment stream, as computed in (3.1), is $\tau_0 = 11.32$ years.

Table 2: Base case parameters.

Parameter	Estimate
Interest rate process	
a	0.0395
b	0.0369
$\overline{\sigma}_r$	0.0195
Price index process	
φ	0.0357
$\overline{\sigma}_\Phi$	0.0081
Stock value process	
$\overline{\sigma}_S$	0.1468
Correlation parameters	
$\rho_{r\Phi}$	-0.0032
ρ_{Sr}	-0.0845
$\rho_{S\Phi}$	-0.0678
Risk premium parameters	
λ_r	-0.2747
λ_Φ	0
λ_S	0.343

This table contains parameter values for interest rate, price index and stock return processes. These parameter values, as well as the price for interest rate risk, are borrowed from from Munk et al. (2004), while the equity risk premium parameter is taken from Brennan and Xia (2002) and the inflation risk premium is set to zero.

Table 3: Distribution of the final funding ratio when utility is from terminal funding ratio and no lower bound is imposed.

$\boldsymbol{\gamma = 2}$			
T	1	10	20
Min	0.58	0.25	0.16
2.5%	0.74	0.52	0.51
25%	0.92	1.03	1.33
50%	1.03	1.46	2.16
75%	1.16	2.1	3.61
97.5%	1.45	4.18	9.37
Max	1.88	9.58	30.22
Mean	1.05	1.68	2.84
St. Dev.	0.18	0.94	2.41
$\mathbb{P}(F_T^{*u} < 1)$	0.42	0.23	0.14
$\mathbb{E}(1 - F_T^{*u} \| F_T^{*u} < 1)$	0.11	0.24	0.28
$\mathbb{E}(F_T^{*u} \| k \leq F_T^{*u})$	1.11	1.89	3.12
$\mathbb{E}(F_T^{*u} \| k \leq F_T^{*u} \leq 1.1)$	1	1	1
$\mathbb{E}(F_T^{*u} \| k \leq F_T^{*u} \leq 1.3)$	1.07	1.1	1.1
$\boldsymbol{\gamma = 5}$			
T	1	10	20
Min	0.8	0.58	0.49
2.5%	0.89	0.78	0.79
25%	0.97	1.02	1.15
50%	1.01	1.18	1.4
75%	1.06	1.36	1.72
97.5%	1.16	1.79	2.51
Max	1.29	2.5	4.01
Mean	1.01	1.2	1.46
St. Dev.	0.07	0.25	0.44
$\mathbb{P}(F_T^{*u} < 1)$	0.43	0.22	0.12
$\mathbb{E}(1 - F_T^{*u} \| F_T^{*u} < 1)$	0.05	0.11	0.13
$\mathbb{E}(F_T^{*u} \| k \leq F_T^{*u})$	1.02	1.24	1.51
$\mathbb{E}(F_T^{*u} \| k \leq F_T^{*u} \leq 1.1)$	1.01	1.01	1.01
$\mathbb{E}(F_T^{*u} \| k \leq F_T^{*u} \leq 1.3)$	1.02	1.11	1.12
$\boldsymbol{\gamma = 10}$			
T	1	10	20
Min	0.89	0.75	0.69
2.5%	0.94	0.87	0.87
25%	0.98	1	1.05
50%	1	1.07	1.15
75%	1.03	1.15	1.28
97.5%	1.07	1.32	1.55
Max	1.13	1.56	1.96
Mean	1	1.08	1.17
St. Dev.	0.03	0.11	0.17
$\mathbb{P}(F_T^{*u} < 1)$	0.47	0.26	0.16
$\mathbb{E}(1 - F_T^{*u} \| F_T^{*u} < 1)$	0.03	0.06	0.07
$\mathbb{E}(F_T^{*u} \| k \leq F_T^{*u})$	1	1.09	1.18
$\mathbb{E}(F_T^{*u} \| k \leq F_T^{*u} \leq 1.1)$	1	1.02	1.02
$\mathbb{E}(F_T^{*u} \| k \leq F_T^{*u} \leq 1.3)$	1	1.08	1.12

This table reports the minimum and the maximum of the distribution of the terminal funding ratio, the 2.5%, 25%, 50%, 75% and 97.5% quantiles, the mean and the standard deviation. Also reported are the shortfall probability, the expected shortfall and the conditional mean of the funding ratio given it lies above $k = 0.9$, or between 0.9 and 1.1, or between 0.9 and 1.3. Parameters are fixed at their base case values (see table 2). Several liability payments take place and we take $T_0 = 75$ years.

Table 4: Present value of contributions and certainty equivalents for different values of δt and m when the pension fund has utility function u.

	$\gamma = 2$								
δt	1				3				10
m	1	3	5	10	1	3	5	10	1
$\frac{C_0(\delta t,m,A_0)}{A_0}$ (%)	28.184	21.246	19.209	17.49	25.308	19.813	18.264	16.986	15.714
$\frac{a_{\text{eq}}(\delta t,m,A_0)}{A_0}$ (%)	11.175	5.675	3.846	2.121	8.174	3.984	2.685	1.462	0
$\frac{C_0(T,1,A_0+a_{\text{eq}}(\delta t,m,A_0))}{A_0}$ (%)	12.403	13.948	14.497	15.03	13.225	14.455	14.853	15.239	15.714
$\frac{\Delta}{A_0}$ (%)	−4.607	−1.623	−0.866	−0.339	−3.909	−1.374	−0.726	−0.284	0

	$\gamma = 5$								
δt	1				3				10
m	1	3	5	10	1	3	5	10	1
$\frac{C_0(\delta t,m,A_0)}{A_0}$ (%)	6.325	4.795	4.378	4.012	5.765	4.548	4.218	3.928	3.602
$\frac{a_{\text{eq}}(\delta t,m,A_0)}{A_0}$ (%)	2.168	1.044	0.703	0.385	1.665	0.756	0.505	0.275	0
$\frac{C_0(T,1,A_0+a_{\text{eq}}(\delta t,m,A_0))}{A_0}$ (%)	3.065	3.335	3.421	3.502	3.184	3.408	3.471	3.53	3.602
$\frac{\Delta}{A_0}$ (%)	−1.092	−0.415	−0.254	−0.124	−0.916	−0.384	−0.241	−0.123	0

	$\gamma = 10$								
δt	1				3				10
m	1	3	5	10	1	3	5	10	1
$\frac{C_0(\delta t,m,A_0)}{A_0}$ (%)	1.181	0.872	0.794	0.729	1.043	0.818	0.76	0.709	0.657
$\frac{a_{\text{eq}}(\delta t,m,A_0)}{A_0}$ (%)	0.326	0.141	0.09	0.047	0.221	0.09	0.058	0.028	0
$\frac{C_0(T,1,A_0+a_{\text{eq}}(\delta t,m,A_0))}{A_0}$ (%)	0.615	0.639	0.645	0.651	0.629	0.645	0.649	0.653	0.657
$\frac{\Delta}{A_0}$ (%)	−0.24	−0.092	−0.058	−0.031	−0.194	−0.083	−0.053	−0.028	0

This table reports the present value of contributions, $C_0(\delta t, m, A_0)$, when the initial asset is A_0, the funding ratio is checked every δt years and the sponsor can split contributions into m parts, and the present value of contributions, $C_0(T, 1, A_0 + a_{\text{eq}}(\delta t, m, A_0))$, when the initial asset is $A_0 + a_{\text{eq}}(\delta t, m, A_0)$, the funding ratio is checked only at time T and the sponsor has to fill the gap immediately. $a_{\text{eq}}(\delta t, m, A_0)$ denotes the certainty equivalent, such that strategies (a) and (b) lead to the same expected utilities. Δ is defined as $a_{\text{eq}}(\delta t, m, A_0) + C_0(T, 1, A_0 + a_{\text{eq}}(\delta t, m, A_0)) - C_0(\delta t, m, A_0)$. Unless otherwise stated, parameters are fixed at their base case values (see table 2), and the initial funding of the pension fund is 1, i.e. $A_0 = L_0 = 0.797$. The minimum legal funding k is set at 0.9. The date of the single payment is $T_0 = 11.32$ years and the time horizon of the fund is $T = 10$ years.

Table 5: Distribution of the final funding ratio when utility is from terminal funding ratio and an explicit lower bound is imposed.

	$\gamma = 2$		
T	1	10	20
Min	0.9	0.9	0.9
2.5%	0.9	0.9	0.9
25%	0.9	0.9	0.9
50%	0.99	0.98	0.93
75%	1.11	1.4	1.56
97.5%	1.39	2.8	4.04
Max	1.81	6.41	13.04
Mean	1.03	1.24	1.39
St. Dev.	0.14	0.54	0.93
$\mathbb{P}(F_T^{*k} < 1)$	0.52	0.52	0.54
$\mathbb{E}(1 - F_T^{*k} \| F_T^{*k} < 1)$	0.08	0.09	0.09
$\mathbb{E}(F_T^{*k} \| k \leq F_T^{*k})$	1.03	1.24	1.39
$\mathbb{E}(F_T^{*k} \| k \leq F_T^{*k} \leq 1.1)$	0.96	0.93	0.92
$\mathbb{E}(F_T^{*k} \| k \leq F_T^{*k} \leq 1.3)$	1.01	0.97	0.95
	$\gamma = 5$		
T	1	10	20
Min	0.9	0.9	0.9
2.5%	0.9	0.9	0.9
25%	0.96	0.93	0.9
50%	1.01	1.07	1.07
75%	1.06	1.23	1.32
97.5%	1.16	1.63	1.93
Max	1.28	2.27	3.1
Mean	1.01	1.11	1.16
St. Dev.	0.07	0.21	0.3
$\mathbb{P}(F_T^{*k} < 1)$	0.44	0.37	0.4
$\mathbb{E}(1 - F_T^{*k} \| F_T^{*k} < 1)$	0.05	0.08	0.08
$\mathbb{E}(F_T^{*k} \| k \leq F_T^{*k})$	1.01	1.11	1.16
$\mathbb{E}(F_T^{*k} \| k \leq F_T^{*k} \leq 1.1)$	1	0.97	0.95
$\mathbb{E}(F_T^{*k} \| k \leq F_T^{*k} \leq 1.3)$	1.01	1.04	1.02
	$\gamma = 10$		
T	1	10	20
Min	0.9	0.9	0.9
2.5%	0.94	0.9	0.9
25%	0.98	0.97	0.95
50%	1	1.05	1.05
75%	1.03	1.12	1.16
97.5%	1.07	1.29	1.41
Max	1.13	1.52	1.78
Mean	1	1.05	1.07
St. Dev.	0.03	0.1	0.14
$\mathbb{P}(F_T^{*k} < 1)$	0.47	0.34	0.37
$\mathbb{E}(1 - F_T^{*k} \| F_T^{*k} < 1)$	0.03	0.06	0.07
$\mathbb{E}(F_T^{*k} \| k \leq F_T^{*k})$	1	1.05	1.07
$\mathbb{E}(F_T^{*k} \| k \leq F_T^{*k} \leq 1.1)$	1	1	0.98
$\mathbb{E}(F_T^{*k} \| k \leq F_T^{*k} \leq 1.3)$	1	1.05	1.05

This table reports the minimum and the maximum of the distribution of the terminal funding ratio, the 2.5%, 25%, 50%, 75% and 97.5% quantiles, the mean and the standard deviation. Also reported are the shortfall probability, the expected shortfall and the conditional mean of the funding ratio given that it lies between $k = 0.9$ and 1.1 or 1.3. Parameters are fixed at their base case values (see table 2). The explicit lower bound k is set to 0.9. Several liability payments take place and we have $T_0 = 75$ years.

Table 6: Distribution of the final funding ratio when utility is from terminal funding ratio and explicit lower and upper bounds are imposed ($k' = 1.1$).

	$\gamma = 2$		
T	1	10	20
Min	0.9	0.9	0.9
2.5%	0.9	0.9	0.9
25%	0.93	0.95	0.9
50%	1.04	1.1	1.1
75%	1.1	1.1	1.1
97.5%	1.1	1.1	1.1
Max	1.1	1.1	1.1
Mean	1.02	1.04	1.04
St. Dev.	0.08	0.09	0.09
$\mathbb{P}(F_T^{*k,k'} < 1)$	0.41	0.28	0.31
$\mathbb{E}(1 - F_T^{*k,k'} \| F_T^{*k,k'} < 1)$	0.07	0.09	0.09
$\mathbb{E}(F_T^{*k,k'} \| k \leq F_T^{*k,k'})$	1.02	1.04	1.04
$\mathbb{E}(F_T^{*k,k'} \| k \leq F_T^{*k,k'} \leq 1.1)$	1.02	1.04	1.04
$\mathbb{E}(F_T^{*k,k'} \| k \leq F_T^{*k,k'} \leq 1.3)$	1.02	1.04	1.04
	$\gamma = 5$		
T	1	10	20
Min	0.9	0.9	0.9
2.5%	0.9	0.9	0.9
25%	0.97	0.97	0.95
50%	1.01	1.1	1.1
75%	1.06	1.1	1.1
97.5%	1.1	1.1	1.1
Max	1.1	1.1	1.1
Mean	1.01	1.04	1.04
St. Dev.	0.06	0.08	0.08
$\mathbb{P}(F_T^{*k,k'} < 1)$	0.43	0.29	0.32
$\mathbb{E}(1 - F_T^{*k,k'} \| F_T^{*k,k'} < 1)$	0.05	0.07	0.08
$\mathbb{E}(F_T^{*k,k'} \| k \leq F_T^{*k,k'})$	1.01	1.04	1.04
$\mathbb{E}(F_T^{*k,k'} \| k \leq F_T^{*k,k'} \leq 1.1)$	1.01	1.04	1.04
$\mathbb{E}(F_T^{*k,k'} \| k \leq F_T^{*k,k'} \leq 1.3)$	1.01	1.04	1.04
	$\gamma = 10$		
T	1	10	20
Min	0.9	0.9	0.9
2.5%	0.94	0.9	0.9
25%	0.98	0.98	0.96
50%	1	1.05	1.06
75%	1.03	1.1	1.1
97.5%	1.07	1.1	1.1
Max	1.1	1.1	1.1
Mean	1	1.03	1.03
St. Dev.	0.03	0.07	0.08
$\mathbb{P}(F_T^{*k,k'} < 1)$	0.47	0.32	0.34
$\mathbb{E}(1 - F_T^{*k,k'} \| F_T^{*k,k'} < 1)$	0.03	0.05	0.07
$\mathbb{E}(F_T^{*k,k'} \| k \leq F_T^{*k,k'})$	1	1.03	1.03
$\mathbb{E}(F_T^{*k,k'} \| k \leq F_T^{*k,k'} \leq 1.1)$	1	1.03	1.03
$\mathbb{E}(F_T^{*k,k'} \| k \leq F_T^{*k,k'} \leq 1.3)$	1	1.03	1.03

This table reports the minimum and the maximum of the distribution of the terminal funding ratio, the 2.5%, 25%, 50%, 75% and 97.5% quantiles, the mean and the standard deviation. Also reported are the shortfall probability and the expected shortfall and the conditional mean of the funding ratio given that it lies between $k = 0.9$ and 1.1 or 1.3. Parameters are fixed at their base case values (see table 2). Several liability payments take place and we have $T_0 = 75$ years. The explicit lower bound k is set to 0.9 and the explicit upper bound k' at 1.1.

Table 7: Comparison of the costs of a strategy involving contributions and of a strategy with explicit lower bound.

	$\gamma = 2$								
δt	1				3				10
m	1	3	5	10	1	3	5	10	1
$\frac{C_0(\delta t, m, A_0)}{A_0}$ (%)	28.184	21.246	19.209	17.49	25.308	19.813	18.264	16.986	15.714
$\frac{a^c_{eq}(\delta t, m, A_0)}{A_0}$ (%)	23.577	19.623	18.343	17.151	21.399	18.439	17.539	16.702	15.714
$\frac{C^c_0}{A_0}$ (%)	0	0	0	0	0	0	0	0	0
$\frac{\Delta^c}{A_0}$ (%)	−4.607	−1.623	−0.866	−0.339	−3.909	−1.374	−0.726	−0.284	0
	$\gamma = 5$								
δt	1				3				10
m	1	3	5	10	1	3	5	10	1
$\frac{C_0(\delta t, m, A_0)}{A_0}$ (%)	6.325	4.795	4.378	4.012	5.765	4.548	4.218	3.928	3.602
$\frac{a^c_{eq}(\delta t, m, A_0)}{A_0}$ (%)	5.233	4.38	4.124	3.887	4.849	4.164	3.977	3.806	3.602
$\frac{C^c_0}{A_0}$ (%)	0	0	0	0	0	0	0	0	0
$\frac{\Delta^c}{A_0}$ (%)	−1.092	−0.415	−0.254	−0.124	−0.916	−0.384	−0.241	−0.123	0
	$\gamma = 10$								
δt	1				3				10
m	1	3	5	10	1	3	5	10	1
$\frac{C_0(\delta t, m, A_0)}{A_0}$ (%)	1.181	0.872	0.794	0.729	1.043	0.818	0.76	0.709	0.657
$\frac{a^c_{eq}(\delta t, m, A_0)}{A_0}$ (%)	0.942	0.779	0.736	0.698	0.849	0.735	0.707	0.681	0.657
$\frac{C^c_0}{A_0}$ (%)	0	0	0	0	0	0	0	0	0
$\frac{\Delta^c}{A_0}$ (%)	−0.24	−0.092	−0.058	−0.031	−0.194	−0.083	−0.053	−0.028	0

This table reports the present value of contributions, $C_0(\delta t, m, A_0)$, when the initial asset is A_0, the funding ratio is checked every δt years and the sponsor can split contributions into m parts, and the present value of contributions, C^c_0, when the pension fund implements the constrained strategy (c). $a^c_{eq}(\delta t, m, A_0)$ denotes the certainty equivalent, such that strategies (a) and (c) lead to the same expected utilities. Δ^c is defined as $a^c_{eq}(\delta t, m, A_0) + C^c_0 - C_0(\delta t, m, A_0)$. Unless otherwise stated, parameters are fixed at their base case values (see table 2), and the initial funding of the pension fund is 1, i.e. $A_0 = L_0 = 0.797$. The explicit lower bound k, which is also the minimum legal funding, is set at 0.9. The date of the single payment is $T_0 = 11.32$ years and the time horizon of the fund is $T = 10$ years.

Table 8: Comparison of the costs of a strategy involving contributions and of a strategy with explicit lower and upper bounds.

					$k' = 1.1$				
δt	1				3				10
m	1	3	5	10	1	3	5	10	1
$\frac{C_0(\delta t,m,A_0)}{A_0}$ (%)	28.184	21.246	19.209	17.49	25.308	19.813	18.264	16.986	15.714
$\frac{a^c_{\mathrm{eq}}(\delta t,m,A_0)}{A_0}$ (%)	−1.925	−3.471	−4.058	−4.636	−2.626	−3.925	−4.391	−4.824	−5.363
$\frac{C^c_0}{A_0}$ (%)	0	0	0	0	0	0	0	0	0
$\frac{\Delta^c}{A_0}$ (%)	−30.11	−24.717	−23.267	−22.126	−27.935	−23.738	−22.655	−21.81	−21.077
					$k' = 1.3$				
δt	1				3				10
m	1	3	5	10	1	3	5	10	1
$\frac{C_0(\delta t,m,A_0)}{A_0}$ (%)	28.184	21.246	19.209	17.49	25.308	19.813	18.264	16.986	15.714
$\frac{a^c_{\mathrm{eq}}(\delta t,m,A_0)}{A_0}$ (%)	5.933	3.361	2.461	1.591	4.716	2.631	1.949	1.304	0.525
$\frac{C^c_0}{A_0}$ (%)	0	0	0	0	0	0	0	0	0
$\frac{\Delta^c}{A_0}$ (%)	−22.251	−17.885	−16.748	−15.899	−20.593	−17.182	−16.316	−15.682	−15.19
					$k' = 1.5$				
δt	1				3				10
m	1	3	5	10	1	3	5	10	1
$\frac{C_0(\delta t,m,A_0)}{A_0}$ (%)	28.184	21.246	19.209	17.49	25.308	19.813	18.264	16.986	15.714
$\frac{a^c_{\mathrm{eq}}(\delta t,m,A_0)}{A_0}$ (%)	11.253	8.079	6.992	5.959	9.69	7.173	6.37	5.607	4.709
$\frac{C^c_0}{A_0}$ (%)	0	0	0	0	0	0	0	0	0
$\frac{\Delta^c}{A_0}$ (%)	−16.931	−13.167	−12.216	−11.532	−15.618	−12.64	−11.894	−11.379	−11.005
					$k' = 1.7$				
δt	1				3				10
m	1	3	5	10	1	3	5	10	1
$\frac{C_0(\delta t,m,A_0)}{A_0}$ (%)	28.184	21.246	19.209	17.49	25.308	19.813	18.264	16.986	15.714
$\frac{a^c_{\mathrm{eq}}(\delta t,m,A_0)}{A_0}$ (%)	14.93	11.388	10.191	9.073	13.144	10.366	9.497	8.681	7.704
$\frac{C^c_0}{A_0}$ (%)	0	0	0	0	0	0	0	0	0
$\frac{\Delta^c}{A_0}$ (%)	−13.254	−9.859	−9.018	−8.417	−12.164	−9.447	−8.767	−8.304	−8.010
					$k' = 1.9$				
δt	1				3				10
m	1	3	5	10	1	3	5	10	1
$\frac{C_0(\delta t,m,A_0)}{A_0}$ (%)	28.184	21.246	19.209	17.49	25.308	19.813	18.264	16.986	15.714
$\frac{a^c_{\mathrm{eq}}(\delta t,m,A_0)}{A_0}$ (%)	17.466	13.703	12.445	11.255	15.523	12.602	11.689	10.831	9.821
$\frac{C^c_0}{A_0}$ (%)	0	0	0	0	0	0	0	0	0
$\frac{\Delta^c}{A_0}$ (%)	−10.718	−7.544	−6.764	−6.235	−9.785	−7.211	−6.576	−6.154	−5.894

This table reports the present value of contributions, $C_0(\delta t, m, A_0)$, when the initial asset is A_0, the funding ratio is checked every δt years and the sponsor can split contributions into m parts, and the present value of contributions, C^c_0, when the pension fund implements the constrained strategy (c). $a^c_{\mathrm{eq}}(\delta t, m, A_0)$ denotes the certainty equivalent, such that strategies (a) and (c) lead to the same expected utilities. Δ^c is defined as $a_{\mathrm{eq}}(\delta t, m, A_0) + C^c_0 - C_0(\delta t, m, A_0)$. Unless otherwise stated, parameters are fixed at their base case values (see table 2), and the initial funding of the pension fund is 1, i.e. $A_0 = L_0 = 0.797$. The explicit lower bound k, which is also the minimum legal funding, is set at 0.9 and the risk aversion parameter γ at 2. The date of the single payment is $T_0 = 11.32$ years and the time horizon of the fund is $T = 10$ years.