Valuation of pension liabilities in incomplete markets*

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Abstract

This paper discusses the valuation of wage-indexed pension liabilities. Valuation of these contingent claims by replication is typically not possible as the wage index cannot be hedged perfectly with financial market instruments. This paper discusses several methods to find a value in such incomplete markets and advocates utility-based valuation. This approach implies a simple adjustment on the discount factor.

Keywords: incomplete markets, pension valuation, wage risk.

JEL codes: G13, H55

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1 Introduction

The valuation of liabilities of pension funds is in a state of flux. Traditional actuarial rules prescribe discounting of the expected cash flows at a fixed interest rate (typically 4%), although it is not always clear whether this should be applied to the nominal or indexed pension claims. Recently, due to changes in the supervisory and accounting rules, pension funds are required to value their liabilities at market prices. Typically, this is done by discounting the liabilities with a market interest rate for loans with the same maturity and risk characteristics. In practice, this approach poses some challenges, as pension fund liabilities are typically not marketed assets. The long maturity of the claims and, perhaps most importantly, the indexation to price or wage growth make it impossible to find a portfolio of perfectly matching market instruments. Therefore, finding the right discount rate is problematic.

In the literature on financial asset pricing (see e.g. Duffie, 1996), the situation where a payoff cannot be replicated perfectly using market instruments is referred to as a situation of an incomplete market. In such a setting, part of the payoff risk on the derivative is uncorrelated to the price changes in the replicating assets. Reasons why replication may be imperfect are the lack of traded instruments, infrequent (non-continuous) trading and transaction costs. For the case of pension liabilities, the major source of market incompleteness is the absence of financial market instruments that exactly replicate the price or wage index that the pensions are linked to. Although there may be products that correlate strongly with the price level, such as index linked bonds, this is definitely not the case for wage-linked payoffs. Hence, for the valuation of wage linked pension liabilities one cannot use a replicating argument directly.

The academic literature offers a few methods for the pricing of assets in incomplete markets. A first approach is to find a super-replicating portfolio, see e.g. Cvitanic, Pham and Touzi (1998, 1999). This is a portfolio whose payoffs are always (in any state of the world) at least as big as the payoff of the derivative. The value of the derivative is then bounded by the value of the super-replicating portfolio. Although for some products, such as options, this may be a useful approach, for pension fund liabilities this approach is not very helpful as the unhedgeable wage growth is in principle unbounded and no super-replicating portfolio can be found.

A second approach, developed by Cochrane and Saá-Requejo (2000) weakens the no-arbitrage argument and finds arbitrage strategies that have bounded Sharpe ratio's. Cochrane and Saá-Requejo claim that even in incomplete markets, this argument may lead to very tight bounds on option values.

The third approach that we consider is utility-based valuation. The investor in the pension liability assumes unhedgeable risk, which will affect the probability distribution of his consumption and final wealth level, and hence his utility. The certainty equivalent wealth of the expected utility then is the value an investor is prepared to pay for the claim. Conversely, we may ask the question how much wealth the pension fund should invest in the market to give their members the same utility as a fully guaranteed wage indexed pension. This wealth then is the "shadow" market value of the (partially) unhegdeable claim. This approach is an application of the utility undifference pricing methodology, introduced in dynamic asset allocation problems by Hodges and Neuberger (1989) and surveyed by Henderson and Hobson (2004).

The final approach we consider is suggested by He and Pearson (1991) and Sangvinatsos and Wachter (2005). They propose to find a market price of risk that makes the (aggregate) demand for a newly introduced asset that completes the market equal to zero.

In this paper, we discuss these approaches to pricing in incomplete markets in more detail. We also and apply these methods to the specific case of pension liabilities which have unhedgeable wage indexation risk. We demonstrate that finding a value for the pension liability amounts to finding a value for the price of the residual wage risk. We show that this valuation method can be implemented in a simple way by adjusting the discount rate for pension claims. The methods suggested in this paper can also be used to price other financial instruments whose payoff is linked to an index. Examples are TIPS, GDP bonds and longevity bonds.

The organization of this paper is as follows. Section 2 provides a detailed introduction to the various valuation principles for incomplete markets. Section 3 applies these methods to the case of wage linked assets and pension claims. Section 4 calibrates the valuation model to data from the Netherlands and the US. Finally, Section 5 draws some conclusions.

2 Valuation principles

The workhorse of asset pricing theory is the pricing kernel or stochastic discount factor.[1] This is a random variable that can be used to price any asset in the economy. The price of a contingent claim is given as[2]

$$P_0 = \mathsf{E}\left[\sum_{t=1}^{T} M_t X_t\right] \qquad (1)$$

[1] See Chochrane (2001) for an introduction to stochastic discount factors.

[2] Unless otherwise indicated, all expectations are as of time zero ($t = 0$).

where X_t are the (random) cash flows generated by the claim at time t and M_t is the pricing kernel. It can be demonstrated that such a pricing kernel exists in any economy where the law of one price holds. If we also assume absence of arbitrage, the pricing kernel is always positive, i.e. a positive payoff will always have a positive price. The pricing kernel is not necessarily unique; there may be many M's that give the correct prices. Only in the case where markets are complete the pricing kernel is unique.

An important example of a complete markets setting is the Black and Scholes (1973) asset pricing model. In this model, the price of the basic asset (a stock) follows a geometric Brownian motion with expected return μ and volatility σ:

$$dS/S = \mu dt + \sigma dZ \tag{2}$$

In addition to this stock, there exists a risk free asset with constant return r. Cochrane (2001) shows that the pricing kernel in this model is

$$M_t = \exp(-rt)\exp\left(-\tfrac{1}{2}\lambda^2 - \lambda Z_t\right) \tag{3}$$

where Z_t is the Brownian motion driving the stock price, and λ is the Sharpe ratio of the stock

$$\lambda = \frac{\mu - r}{\sigma} \tag{4}$$

The interpretation for this pricing kernel is simple. The first part is the risk-free discount factor, and the second part is random variable which puts a high weight on payoffs that occur at low values of the stock (i.e. low values of Z_t). The pricing kernel depends on the risk-free interest rate, which captures the time value of money, and the Sharpe ratio of the stock, which captures the risk-return tradeoff for stock market exposure. This property holds more generally: with multiple sources of risk in the economy, the stochastic discount factor can typically be written as a function of the risk free rate, the Sharpe ratio's of marketed assets and the shocks to the asset prices, and λ and dZ_t are vectors of Sharpe ratio's and stochastic shocks, respectively. The dynamics of the SDF are given by

$$dM_t/M_t = -rdt - \lambda dZ_t \tag{5}$$

With this pricing kernel, any contingent claim with payoff $X_t = f(S_t)$ can be priced by calculating the expectation

$$P_0 = \mathsf{E}[M_t f(S_t)] \tag{6}$$

In the Black-Scholes example, there is only one source of uncertainty, the stock price, and the market is (dynamically) complete, since the payoff of any stock price derivative can be attained by a trading strategy in the stock and the risk-free asset. This is of course a very stylized setting and in practice, the payoff of claims such as pension liabilities depend on multiple of risk factors. Moreover, some of these risk factors cannot be replicated by trading in financial market instruments. How to deal with pricing and valuation in such a situation?

2.1 Good deal bounds

One guideline for the valuation of claims in an incomplete market is given by Cochrane and Saá-Requejo (2000) [CS from now on].[3] Suppose the uncertainty in the payoffs is generated by the stock price from equation (2) and an additional state variable V, whose dynamics can be described by

$$dV_t/V_t = \mu_V + \sigma_V dZ_t + \sigma_{Vw} dZ_t^w \tag{7}$$

where Z_t^w is a Brownian motion that is independent of Z_t. The coefficients σ_V and σ_{Vw} denote the exposure of the state variable V to the stock price shocks and the independent Brownian motion.

CS show that the pricing kernel in this economy has dynamics

$$dM_t/M_t = -rdt - \lambda dZ_t - \lambda_w dZ_t^w \tag{8}$$

where, as before, λ is the Sharpe ratio of the stock and λ_w is an unknown coefficient. In an incomplete market (where there are no traded assets that correlate with Z_w) the no-arbitrage framework doesn't pin down the value of λ_w, so that we're left with an infinity of possible SDF's and, by implication, infinitely many possible values for contingent claims that depend on the value of the state variable V. The insight of CS is that λ_w can be interpreted as the Sharpe ratio of a (hypothetical) asset that has a non-zero exposure to dZ_t^w and no exposure to dZ_t. Now CS suggest to put a bound on the absolute value of this Sharpe ratio, such that too 'good deals', i.e. assets or positions with too high (absolute) Sharpe ratios, are ruled out. The bounds on the Sharpe ratio then imply an upper and lower bound for the value of any contingent claim. Of course, the magnitude of the bound on the Sharpe ratio is arbitrary, but CS show that even fairly generous values for the bound may lead to very tight ranges for the implied prices of derivatives such as options. In Section 3.1 we analyze this method for the case of wage indexed pension claims.

2.2 Utility-based pricing

A alternative approach to pricing in incomplete markets is based on the utility a representative agent can derive from investing in the untraded claim. This approach has been used

[3] Related ideas are presented by Roorda, Schumacher and Engwerda (2005).

extensively in the literature, for example by Svensson and Werner (1993), Henderson (2002, 2005) and Chen, Pelsser and Vellekoop (2007). These papers study a situation where a new derivative is added to the existing portfolio of an investment bank or a new product is added to the portfolio of an insurer, and part of the risk on the new derivative or product cannot be hedged using market instruments. Our analysis builds on these papers and is also closely related to work on asset allocation in incomplete markets, see for example the work by Brennan and Xia (2002), Munk and Sørensen (2007) and De Jong, Driessen and Van Hemert (2007). We now first discuss the general setup of the utility maximization problem and outline how it can be used to find the value for a non-traded claim. In the Section 3 we present a detailed model that is explicitly targeted at valuing long-dated pension claims.

The setting we study is a stylized representation of an individual saving for retirement. The main simplification is that we consider a terminal wealth problem and ignore intermediate consumption and labour income. Thus, the economic agent maximizes expected utility of terminal wealth over all the possible investment strategies. First consider the complete markets problem, where any value of terminal wealth can be attained by an investment strategy in financial market instruments. In that case, the agent solves the problem

$$\max_{W_{1T}} \mathsf{E}\left[U(W_{1T})\right] \quad \text{s.t.} \quad \mathsf{E}[W_{1T}M_{1T}] = W_0 \tag{9}$$

where $U(W)$ is the utility function, W_0 the initial wealth, M_{1T} is the pricing kernel for the marketed risks, and the latter condition is the budget constraint. From the mathematical finance literature (see e.g. Cox and Huang, 1989, and Pliska, 1997) it is well known that the optimal investment strategy is given by

$$W_{1T}^{*} = U'^{-1}\left(\ell_1 M_{1T}\right) \tag{10}$$

where ℓ_1 is the Lagrange multiplier of the budget constraint.

For the incomplete markets setting, in general, finding the optimal investment policy is a hard problem and an optimal wealth strategy cannot be derived using the Cox-Huang method. However, we assume a particular structure on the utility function and the risk, such that the problem can be simplified enough to apply these methods. First, assume that terminal wealth can be decomposed as the product of a perfectly hedgeable wealth component W_{1T} and an independent component W_{2T}. The additional term W_{2T} can be interpreted as exogenous background risk; here it is assumed that the stochastic processes W_1 and W_2 are independent. The agent then solves the problem over all feasible values for W_T, subject to a budget constraint

$$\max_{W_{1T}} \mathsf{E}\left[U(W_{1T}W_{2T})\right] \quad \text{s.t.} \quad \mathsf{E}[W_{1T}M_{1T}] = W_0 \tag{11}$$

The optimal wealth strategy can be derived from the first order condition

$$h(W_{1T}) \equiv \mathsf{E}\left[U'(W_{1T}W_{2T})W_{2T}|W_{1T}\right] = \mu_1 M_{1T} \tag{12}$$

so that

$$\widehat{W}_{1T} = h^{-1}(\mu_1 M_{1T}) \tag{13}$$

We now have all the ingredients to find the equivalent utility price of the unhedgeable risk. Define the expected utility of complete markets problem as

$$I(W_0) = \mathsf{E}\left[U(W_{1T}^*)\right] \tag{14}$$

With background risk, the expected utility is

$$J(W_0) = \mathsf{E}\left[U(\widehat{W}_{1T}W_{2T})\right] \tag{15}$$

Hence, to obtain the same expected utility as in complete markets case, an additional fraction of initial wealth π has to be invested, with

$$I(W_0) = J(\pi W_0) \tag{16}$$

We work out the value for π for the case of wage indexed pensions in Section 3.2.

2.3 Equilibrium pricing kernel

An alternative way to find the value of a claim in incomplete markets follows an argument by He and Pearson (1991) and Sangvinatsos and Wachter (2005). They suggest to complete the market with a new instrument to hedge the previously unhedged risk and to augment the pricing kernel with the new risk factor. Then choose the additional parameter of the pricing kernel such that the demand for the new instrument equals exactly zero. This pricing kernel parameter will determine the equilibrium market price of risk of the additional risk factor and thus the market price of the new instrument. This approach will applied to wage-indexed pension claims and wage linked bonds in Section 3.3.

3 Wage risk

In this section, we apply the equivalent utility pricing principle to the case of unhedgable real wage risk. We first present a specific model for a long-horizon investor who faces wage growth risk. The objective of the consumer is to maximize the utility of wage-deflated terminal wealth.[4] This is different from the Brennan and Xia (2002) model, where the agent maximizes the utility of price deflated wealth. This is our way of introducing wage risk into the model. The optimization problem is

$$\max \mathsf{E}\left[U(W_T/Y_T)\right] \tag{17}$$

subject to a budget constraint, where W_T denotes undeflated wealth and Y_T denotes the wage level.

The stochastic model for financial market prices and wages is similar to the model of Svensson and Werner (1993). The model has a stock return and wage growth equation. We allow for time varying interest rates. This is important, as pension claims are long-dated and pension funds therefore face large interest rate risks. The equations driving these variables are

$$dS/S = (r + \lambda_S \sigma_S)dt + \sigma_S dZ_S \tag{18a}$$

$$dr = a(\bar{r} - r)dt + \sigma_r dZ_r \tag{18b}$$

$$dY/Y = gdt + \theta' dZ + \theta_w dZ_w \tag{18c}$$

where the vector $Z = (Z_S, Z_r)$ and the covariance matrix of dZ is ρ. In the last equation, Y is the wage index level and g denotes the expected wage growth. The term $\theta' dZ$ denotes the exposure of wage growth to the shocks in stock returns and interest rates. The final term $\theta_w dZ_w$ denotes the idiosyncratic wage risk, which is by assumption uncorrelated with the other risk factors. We assume that in the economy there are stocks and a full menu of bonds, so that interest rate risk can be hedged perfectly, but there are no traded wage linked assets. Asset values are determined from the pricing kernel

$$dM_1/M_1 = -rdt + \phi' dZ \tag{19}$$

where $\phi = -\rho^{-1}\lambda$ and $\lambda = (\lambda_S, \lambda_r)'$ is the vector of market prices of risk or Sharpe ratio's. Brennan and Xia (2002) show that the price dynamics of a T period zero-coupon bond are

$$dP(T)/P(T) = [r - B(T)\sigma_r\lambda_r]dt - B(T)\sigma_r dZ_r \tag{20}$$

with $B(T)$ a function of the time to maturity T of the bond, that follows from the Vasicek (1977) model

$$B(T) = \frac{1 - \exp(-aT)}{a} \tag{21}$$

[4]Brennan and Xia (2002) show that ignoring interim consumption does not affect the results of the optimal investment policy much if the horizon T is chosen appropriately. Ignoring labour income matters more, as this provides an automatic hedge against wage risk. However, analytical solutions for the optimal investment strategy and the utility level do not exist for such a setting. Papers in this literature therefore rely on numerical results instead, see e.g. Munk and Sørensen (2007), and Koijen, Nijman and Werker (2005).

To simplify the analytical results, we assume that all volatilities, market prices of risks and correlations are time-invariant. Hence, interest rates are normal, and stock and bond returns, as well as wages, are log-normal. We furthermore assume that the expected wage growth is constant over time. The model can easily be extended to serially correlated expected wage growth, at the expense of more heavy notation.

3.1 Good deal bounds

The good deal bounds of Cochrane and Saá-Requejo (2000) can be easily applied to the wage risk setting, but as the wage growth indexation applies to the full value of the liabilities, the implied bounds can actually be pretty wide. To give an example, suppose that wages are not correlated with stock prices and interest rates ($\theta = 0$) and the Sharpe ratio of wage-linked assets is bounded by $|\lambda_w| < A$. Then the discount rate to be applied to the payoff of a wage linked bond is bounded between $(r - g - A\theta_w, r - g + A\theta_w)$, where θ_w is the idiosyncratic volatility of wage growth. Estimates in Section 4 show that the interest rate minus the wage growth $(r - g)$ is around 1.5%, and the annualized variance of the idiosyncratic wage growth is 4.5%. We choose $A = 0.2$, which corresponds roughly to the equity market Sharpe ratio.[5] With these numbers, the discount rate is in the range $(0.6\%, 2.4\%)$. Assuming a liability of 100 million euro with a duration of 20 years, this implies a range of present values between 62.23 and 88.72 million, roughly a fifty percent difference. This is a fairly wide range and may be too wide to get good judgements of the solvency of the fund.

3.2 Utility-based valuation of wage risk

For the utility based valuation, we return to the utility maximization problem in equation (17). The optimization problem with explicit budget constraint is

$$\max_{W_T} \mathsf{E}\left[U(W_T/Y_T)\right], \quad \text{s.t. } \mathsf{E}\left[W_T M_{1T}\right] = W_0 \tag{22}$$

where M_{1T} is the pricing kernel for tradeable wealth, denoted by W_T. The wage level is decomposed as $Y_t = Y_{1t}Y_{2t}$, where Y_1 denotes the part of wage growth that can be spanned by traded assets, and Y_2 the orthogonal part.[6] The dynamics of the wage components therefore are

$$dY_1/Y_1 = gdt + \theta' dZ \tag{23a}$$

$$dY_2/Y_2 = \theta_w dZ_w \tag{23b}$$

[5] Here we use an estimate of the equity premium $\mu_S - r$ equal to 4% as in Fama and French (2002), and a stock price volatility σ_S of 20%. Together, this implies a Sharpe ratio $\lambda_S = 0.04/0.20 = 0.2$.

[6] In the notation of Section 2, $W_{1T} = W_T/Y_{1T}$ and $W_{2T} = 1/Y_{2T}$.

According to Brennan and Xia (2002), the optimal wealth strategy W_T^* is independent of Y_{2T} because this is just independent background risk that cannot be hedged and therefore does not affect the optimal wealth strategy.[7] Solving for the optimal wealth process gives

$$W_T^*/Y_{1T} = \ell \left(M_{1T}Y_{1T}\right)^{-1/\gamma} \tag{24}$$

where ℓ is solved from the budget constraint. The indirect utility function therefore can be written as

$$J = \mathsf{E}\left[\left(\frac{W_T^*}{Y_T}\right)^{1-\gamma}\right] = \mathsf{E}\left[\left(\frac{W_T^*}{Y_{1T}}\right)^{1-\gamma}\right]\mathsf{E}\left[\left(Y_{2T}^{-1}\right)^{1-\gamma}\right]/(1-\gamma) \tag{25}$$

Using that $\ln Y_{2T} \sim N(-\frac{1}{2}\theta_w^2 T,\ \theta_w^2 T)$ we find

$$J = I\mathsf{E}\left[\left(\frac{1}{Y_{2T}}\right)^{1-\gamma}\right] = I\exp\left\{\tfrac{1}{2}(1-\gamma)\theta_w^2 T + \tfrac{1}{2}(1-\gamma)^2\theta_w^2 T\right\} \tag{26}$$

with

$$I = \mathsf{E}\left[\left(\frac{W_T^*}{Y_{1T}}\right)^{1-\gamma}\right]/(1-\gamma) \tag{27}$$

Notice that I is the indirect utility of the complete markets ($\theta_w = 0$) problem, which is independent of the parameters for Y_{2T}. The certainty equivalent wealth of the optimal portfolio strategy is

$$W_{CE} = U^{-1}(J) = U^{-1}(I)\exp\left((1-\tfrac{1}{2}\gamma)\theta_w^2 T\right) \tag{28}$$

Hence, to achieve the same certainty equivalent wealth in the incomplete markets case, one needs to invest a fraction $\pi = \exp((1-\frac{1}{2}\gamma)\theta_w^2 T)$ more than in the complete markets case. Notice that $\pi > 1$ if $\gamma > 2$, so only for moderately risk averse investors, the elimination of idiosyncratic wage risk is welfare improving.[8]

This result can be used to price wage linked bonds. According to the previous argument, such a bond is valued by an agent who wants to hedge wage risk. This agent requires a per-period risk premium $(1-\frac{1}{2}\gamma)\theta_w^2$ for the idiosyncratic wage risk; notice that this additional risk premium is negative for $\gamma > 2$, so that relatively risk averse investors are willing to give up expected return to get exposure to the wage risk. The return on a fully indexed pension (or wage linked bond), paying off $Y(T)$ at time T, then must be

$$\frac{dP_Y(T)}{P_Y(T)} = [r - B(T)\sigma_r\lambda_r + \theta'\lambda + (1-\tfrac{1}{2}\gamma)\theta_w^2]dt - B(T)\sigma_r dZ_r + \theta' dZ + \theta_w dZ_w \tag{29}$$

[7] The key for this to hold is the CRRA utility function and the multiplicative nature of the unhedgeable risk. The unhedgeable risk therefore does not influence the marginal utility of the hedgeable wealth component (separability). This will be different for the models of Henderson (2002, 2005) and Chen, Pelsser and Vellekoop (2007), where the utility function is CARA and the risk is additive and therefore does impact the marginal utility of hedgeable wealth.

[8] This result may appear counter-intuitive at first sight, but notice that the additional discounting by $(1/Y_{2T})$ is not a mean-preserving spread on the optimal wealth policy because $\mathsf{E}[1/Y_{2T}] = \exp(\theta_w^2 T) > 1$. Hence, for more risk-tolerant investors, eliminating idiosyncratic wage risk is not welfare improving.

with

$$B(T) = \frac{1 - e^{-aT}}{a} \tag{30}$$

The first part of the expected return, $r - B(T)\sigma_r\lambda_r$, is the yield on a T period non-indexed zero-coupon bond. The term $\theta'\lambda$ reflects the risk premium earned on the 'best' (but imperfect) hedging portfolio of stocks and bonds for the wage linked bond.[9] The remaining term, $(1 - \frac{1}{2}\gamma)\theta_w^2$, is the risk premium for idiosyncratic wage risk.

The expected real return on a wage linked bond is the appropriate discount rate for fully indexed pension claims. From the bond return equation (29), the expected real return can be easily derived as

$$r_Y(T) = \mathsf{E}\left[\frac{d(P_Y(T)/Y)}{P_Y(T)/Y}\right]/dt = r - g - B(T)\sigma_r\tilde{\lambda}_r + \theta'\tilde{\lambda} - \tfrac{1}{2}\gamma\theta_w^2 \tag{31}$$

with $\tilde{\lambda} = \lambda - \rho\theta$ the vector of risk premiums corrected for the correlation with wage growth. The expected real return is lower than the expected (nominal) return on an equivalent non-indexed bond by the expected wage growth g; this is similar to the classic Gordon stock valuation model model, where the discount rate equals the cost of capital minus the expected dividend growth. The expected real return is also different because of the correlations between wage growth and the other risk factors, $\theta'\tilde{\lambda}$, which depends on the risk premium on the other risk factors. Finally, the expected real return has an extra term for the market incompleteness. This term is always negative.

The model of Bodie, Merton and Samuelson (1992) is a special case of this model, where there is no idiosyncratic wage risk i.e. $\theta_w = 0$. Svensson and Werner (1993) show, in a CARA-normal framework, that the discount rate for wage linked claims is

$$r_Y = r - g + \lambda'\theta + \Gamma\theta_w^2 \tag{32}$$

where Γ is the coefficient of absolute risk aversion. The additional discount for wage risk is *positive*, because in their model utility is not scaled by the wage level but by the price index. Hence, a position in a wage linked asset adds to uncertainty instead of providing a hedge against fluctuations in deflated wealth.

3.3 Equilibrium pricing kernel for wage risk

Sangvinatsos and Wachter (2005) suggest to find the equilibrium price of wage risk by completing the market with a new instruments to hedge the previously unhedged wage

[9] This part of the risk premium can also be written as $\beta'(\mu - r)$, where $\beta = \sigma(\sigma'\sigma)^{-1}\theta$ are the OLS regression coefficients of wage growth on the asset returns and $\mu - r = \sigma\lambda$ is the vector of risk premiums earned on the assets with an exposure σ to the risk factors dZ.

risk, and to choose the additional parameters of the pricing kernel such that the aggregate demand for new instruments equals exactly zero. In this section, we work out this approach. The advantage over the utility-based approach is that we can easily derive the equilibrium price in an economy which consists of agents with different preferences or different amounts of human capital.

Introduce an asset that spans the previously unhedged real wage risk. When such an asset exists, the pricing kernel can be extended to

$$dM/M = -rdt + \phi' dZ + \phi_w dZ_w \tag{33}$$

Adapting the results of Brennan and Xia (2002) and De Jong (2008), the optimal portfolio weights are given as

$$x^* = (\sigma\bar{\rho}\sigma')^{-1}\sigma\bar{\rho}\left[-\frac{1}{\gamma}\begin{pmatrix}\phi \\ \phi_w\end{pmatrix} + \left(1-\frac{1}{\gamma}\right)\begin{pmatrix}\theta - B(T)\sigma_r e_r \\ \theta_w\end{pmatrix}\right] \tag{34}$$

where σ is the matrix of exposure coefficients of the assets included in the portfolio problem, and $\bar{\rho}$ is the correlation matrix of $(dZ, dZ_w)'$. The first component of expression for the optimal portfolio weight is the usual mean-variance optimal speculative portfolio. The second component contains the hedges against wage risk and interest rate risk.

To simplify the algebra, consider a hypothetical asset that has a positive exposure to the idiosyncratic real wage risk and no exposure to the other sources of risk. The return on this asset is given by

$$dP_w/P_w = [r + \theta_w\lambda_w]dt + \theta_w dZ_w \tag{35}$$

with $\lambda_w = -\phi_w$. With this wage linked asset as the last asset in the vector x, the weight on the wage-linked instrument in equation (34) simplifies to

$$x_w^* = \theta_w^{-1}\left[-\frac{1}{\gamma}\phi_w + \left(1-\frac{1}{\gamma}\right)\theta_w\right] \tag{36}$$

The principle of Sangvinatsos and Wachter (2005) is to find a value ϕ_w such that the demand for wage linked bonds is zero, hence $x_w^* = 0$. The solution to this equality is

$$\phi_w = (\gamma - 1)\theta_w \tag{37}$$

For $\gamma > 1$, the pricing kernel parameter will be positive, and hence the implied risk premium on the new asset, $\lambda_w = (1-\gamma)\theta_w$, will be negative.

In this complete market, the price of a long maturity wage linked bond (WLB) with maturity T obeys

$$dP_Y(T)/P_Y(T) = [r - B(T)\sigma_r\lambda_r + \theta'\lambda + (1-\gamma)\theta_w^2]dt - B(T)\sigma_r dZ_r + \theta' dZ + \theta_w dZ_w \tag{38}$$

The expected real return of the wage linked bond therefore follows as

$$r_Y(T) = \mathsf{E}\left[\frac{d(P_Y(T)/Y)}{P_Y(T)/Y}\right] = r - g - B(T)\sigma_r\tilde{\lambda}_r + \theta'\tilde{\lambda} - \gamma\theta_w^2 \tag{39}$$

There is a small difference with the result based on the equivalent utility argument, equation (31): the additional discount for idiosyncratic risk is $\gamma\theta_w^2$ and not $\frac{1}{2}\gamma\theta_w^2$. This difference is explained by the fact that under the Sangvinatsos-Wachter argument, the investor optimally chooses a position in WLB's, whereas in the equivalent utility argument, the original position in assets is augmented with WLB's such as to compensate for the utility cost of the unhedged risk.

The argument so far considered a single investor who invests all his wealth in financial assets. In reality, even investors have a fraction of their wealth locked up in human capital. The Appendix shows that the demand for the wage linked asset for an investor with human capital to total wealth ratio h satisfies

$$(1-h)x_w^* = \theta_w^{-1}\left[-\frac{1}{\gamma}(\phi_w + \theta_w) + (1-h)\theta_w\right] \tag{40}$$

In an economy with many agents, setting the aggregate demand for the wage linked asset equal zero and solving for the pricing kernel parameter gives

$$\phi_w + \theta_w = \gamma^*(1-h^*)\theta_w \tag{41}$$

where $1/\gamma^*$ is the wealth-weighted average risk tolerance in the economy and h^* the aggregate human capital to total wealth ratio. The intuition for this result is quite simple. The hedge demand for wage linked assets is positive for every agent. In equilibrium with a zero net supply of wage linked assets, this hedge demand has to be compensated with a negative speculative demand, and therefore $\phi_w + \theta_w$ is positive in equilibrium. With this equilibrium risk premium, investors with high risk aversion or low human capital (typically these are the older investors) will have a positive net demand for the wage linked asset, whereas investors with low risk aversion or high human capital will short the wage linked asset. The equilibrium discount rate for wage linked bonds then follows as

$$r_Y(T) = r - g - B(T)\sigma_r\tilde{\lambda}_r + \theta'\tilde{\lambda} - \gamma^*(1-h^*)\theta_w^2 \tag{42}$$

4 Calibration

In this section, we calibrate the model with real wage growth to data from the Netherlands and the US.

4.1 Netherlands

For the Netherlands, the sample period is 1950-2002 (annual data), obtained from Statistics Netherlands. Real wage growth is constructed as nominal wage growth minus realized inflation, and is graphed in Figure 1. The wage growth fluctuates quite a bit and also exhibits some persistence, i.e. years with high wage growth are more likely to be followed by another year of high wage growth. The mean and standard deviation of annual wage growth are 1.5% and 2.65%, respectively. The sum of the coefficients in an AR(3) model (selected by the Akaike information criterion) is 0.55, indicating some persistence in the wage growth process. As this persistence is not explicitly taken into account in the model of this paper, we correct the standard deviation of wage growth to correspond to the annualized standard deviation of the 30-year wage growth. For this, we use the formulas of the variance ratio in Campbell, Lo and MacKinlay (1997, p.49). Corrected for serial correlation, the annualized standard deviation of wage growth is 4.5%.

The exposure coefficients θ_S and θ_r are estimated using the residual covariance matrix of a VAR(1) model for excess stock returns, the ex-post real interest rate (constructed as the difference between the nominal short rate and inflation) and the real wage growth. The estimates are reported in Table 1. These estimates imply a positive but small exposure of wage growth to stock returns and interest rates. Using an equity Sharpe ratio of 0.2, the stock market exposure translates into a risk premium of $\theta_S \lambda_S = 0.0030 * 0.20 = 0.0006$, or only 6 basis points, which is almost negligible.[10] The real interest rate exposure is also small and leads to a negligible addition to the risk premium.

To give a numerical example, assume a risk aversion $\gamma = 5$ and an idiosyncratic wage risk $\theta_w = 4.5\%$, based on the estimate of real wage volatility corrected for serial correlation. We assume a real interest rate of 3% and an expected real wage growth of 1.5%. The discount rate for wage-indexed pensions then is

$$\begin{aligned} r^Y &= r - g + \theta'\lambda - \tfrac{1}{2}\gamma\theta_w^2 \\ &= 3.0\% - 1.5\% + 0\% - 0.5 * 5 * (0.045)^2 \\ &= 1.5\% - 0.5\% = 1.0\% \end{aligned} \tag{43}$$

The additional term caused by the market incompleteness is -0.5%, which seems small, but with such low interest rates it has a significant impact on the valuation. The shadow market value of the 100 million liability from the earlier example will be 81.95 million. Ignoring the

[10] Benzoni, Collin-Dufresne and Goldstein (2007) argue that the long-horizon correlation between stock returns and wages may be a lot higher than the short horizon correlations. Including their cointegration setup in our model would seriously complicate the analysis, however, and we reserve this for future research.

extra term of the market incompleteness (i.e. assuming a discount rate of 1.5%), the value is 74.25, which makes about a 10% difference.

4.2 US

Next, we calibrate the model to US data. As wages in the US exhibit more nominal rigidity than in Europe, the model is estimated including inflation and nominal t-bill rate as additional determinants. Moreover, we include the dividend-price ratio, whcih is a well-known predictor of stock returns. Finally, we include the logarithm of the dividend to wage ratio this follows follow Benzoni, Collin-Dufresne and Goldstein (2007) who use this variable as a predictor for wage growth.

The wage data (Wage and Salary Disbursements) are from the National Income and Product Accounts (NIPA), collected by the Bureau of Economic Analysis of the Department of Commerce. Other data are stock returns (value weighted index including dividends), interest rates (annual returns on 90-day t-bills), CPI inflation and annual dividends (all these variables are obtained from CRSP). Following Heaton and Lucas (2000), we use postwar data, starting in 1946; our sample ends in 2006.

To estimate the long-run variances of these variables and their correlations with wage growth, we follow Campbell and Viceira (2005) and estimate a Vector Autoregression on stock returns, interest rates, inflation, and nominal wage growth. The VAR estimates are reported in Table 2, and Table 3 reports the implied long-run variances and covariances, calculated using the expressions in Hoevenaars et al. (2007). The estimates imply a long-run standard deviation of 4.5% for real wage growth, with positive exposures to stock market returns, a small exposure to inflation, and a negative exposure to interest rates. The magnitude of the exposures is most easily discussed using the beta's of the wage growth, which are obtained form a hypothetical multivariate regression of the long-run wage growth shocks on (long run shocks to) stock returns, interest rates and inflation. The stock return beta is 0.187. The inflation beta is close to zero and the (nominal) interest rate beta is -0.25. This may indicate that even in the long run there is some nominal rigidity in wages. The long-run annual standard deviation of the idiosyncratic wage growth is 2.4%.

We calculate the risk premiums associated with the various sources of risk. For this exercise, we assume a market price of risk of 0.2 for stock returns, -0.15 for interest rate risk and also -0.15 for inflation risk. The stock market risk contributes 45 basis points per annum to the risk premium of a bond linked to the nominal wage index. The inflation risk premium is negligible, but the interest rate risk premium is around 30 basis points. Finally, the premium for the idiosyncratic wage risk is small, around -15 basis points for an investor with risk aversion $\gamma = 5$. All in all, the risk premium on a wage linked bond is 60 basis points higher than the risk premium on a CPI indexed bond, mainly due to the positive stock market exposure of wages.

5 Conclusion and implications for pension funds

Pension funds with wage indexed liabilities face two issues in valuating their liabilities. The first is that the expected wage growth has to be taken into account. This can be achieved fairly simply by correcting the discount rate for the expected wage growth rate. The second is issue is more complex and arises because wage risk cannot be hedged perfectly. In other words, the pension fund cannot replicate a wage exactly with its asset mix. Although the asset mix may be chosen strategically to include assets that show correlation with wage growth, a non-hedgeable residual wage risk remains. Then, to offer the pension fund members the same expected utility as a perfect wage indexed pension, the fund should invest initially more money (and hence pay out more on average) to compensate for this risk. In this paper, we show that this additional value can be calculated fairly easily by adjusting the discount rate of the future claims. We calibrate that the adjustment for the idiosyncratic wage risk is between -15 and -50 basis points. This effect is counterbalanced by the positive exposure of wage growth to the stock market, however, so the net adjustment to the discount rate will be higher and may even be positive.

A Portfolio demand with human capital

This appendix derives the optimal portfolio for an investor with human capital in a complete markets setting. Define human capital as the present value of the stream of future labor income

$$H_t = \mathsf{E}_t \left[\int_t^T M_s Y_s ds \right] \tag{44}$$

The wage process from equation (18c) is

$$dY/Y = gdt + \bar{\theta}' d\bar{Z} \tag{45}$$

with $\bar{\theta} = (\theta, \theta_w)'$ and $d\bar{Z} = (dZ, dZ_w)'$. With the pricing kernel written as

$$dM/M = -rdt + \bar{\phi}' d\bar{Z} \tag{46}$$

where $\bar{\phi} = (\phi, \phi_w)'$, human capital equals

$$H_t = Y_t \int_t^T \exp\left\{-(r_t(s) - g + \bar{\theta}'\bar{\lambda})(s-t)\right\} ds \tag{47}$$

where $r_t(s)$ is the yield at time t on a bond with time to maturity s, and $\bar{\lambda} = (\lambda, \lambda_w)'$ are the market prices of risk. The dynamics of human capital are then

$$dH/H = [r + \bar{\theta}'\bar{\lambda} - B_H \sigma_r \lambda_r]dt + \bar{\theta}' d\bar{Z} - B_H \sigma_r dZ_r \tag{48}$$

where B_H is the interest rate duration of human capital, defined as

$$B_H = \int_{s=t}^{T} \exp\left\{-(r_t(s) - g + \bar{\theta}'\bar{\lambda})(s-t)\right\}/H_t \cdot B(T-s) \tag{49}$$

In a complete financial market, the optimal wealth strategy of the investor is determined from

$$d \ln G = [..]dt - \frac{1}{\gamma}\bar{\phi}' d\bar{Z} + \left(1 - \frac{1}{\gamma}\right)[-B(T)\sigma_r dZ_r + \bar{\theta}' d\bar{Z}] \tag{50}$$

The actual wealth dynamics for portfolio weights x are given by

$$d \ln W = [..]dt + (1-h)x'\sigma d\bar{Z} + h\left[-B_H \sigma_r dZ_r + \bar{\theta}' d\bar{Z}\right] \tag{51}$$

with $h \equiv \frac{H}{F+H}$ is the fraction of human capital in total wealth. Equating these two expressions gives the optimal portfolio weights

$$(1-h)x^* = (\sigma\bar{\rho}\sigma')^{-1}\sigma\bar{\rho}\left[-\frac{1}{\gamma}(\bar{\phi} + \bar{\theta}) + (1-h)\bar{\theta} - be_r\right] \tag{52}$$

with $b = \left(1 - \frac{1}{\gamma}\right)B(T) - hB_H$. With the asset menu containing the hypothetical wage-linked asset with price dynamics (35), the corresponding element of equation (52) simplifies to

$$(1-h)x_w^* = \theta_w^{-1}\left[-\frac{1}{\gamma}(\phi_w + \theta_w) + (1-h)\theta_w\right] \tag{53}$$

Now consider an economy with $i = 1, .., N$ individuals with risk aversion coefficient γ_i, total wealth W_i and human capital to wealth ratio h_i. The aggregate demand for the wage linked asset then is

$$\sum_i W_i(1-h_i)x_{w,i}^* = \theta_w^{-1}\left[-\sum_i \frac{W_i}{\gamma_i}(\phi_w + \theta_w) + \sum_i W_i(1-h_i)\theta_w\right] \tag{54}$$

Equilibrium with zero net supply of the wage linked asset implies

$$\sum_i \frac{W_i}{\gamma_i}(\phi_w + \theta_w) = \sum_i W_i(1-h_i)\theta_w \tag{55}$$

or, equivalently

$$\phi_w + \theta_w = \gamma^*(1-h^*)\theta_w \tag{56}$$

with

$$\frac{1}{\gamma^*} = \frac{\sum_i W_i/\gamma_i}{\sum_i W_i}, \qquad h^* = \frac{\sum_i h_i W_i}{\sum_i W_i} \tag{57}$$

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Table 1: Selected parameter estimates

This table shows selected estimates of the parameters in the model, obtained from annual data for the Netherlands, 1950-2002. The subindex S refers to the excess stock return, r to the ex-post real interest rate (nominal short rate minus inflation) and w to real wage growth.

σ_S	0.2051	θ_S	0.0035
σ_r	0.0237	θ_r	0.0030
ρ_{Sr}	-0.19	θ_w	0.0248

Figure 1: Real wage growth in the Netherlands

This figure shows annual real wage growth rates in the Netherlands, 1950–2002. Source: Statistics Netherlands.

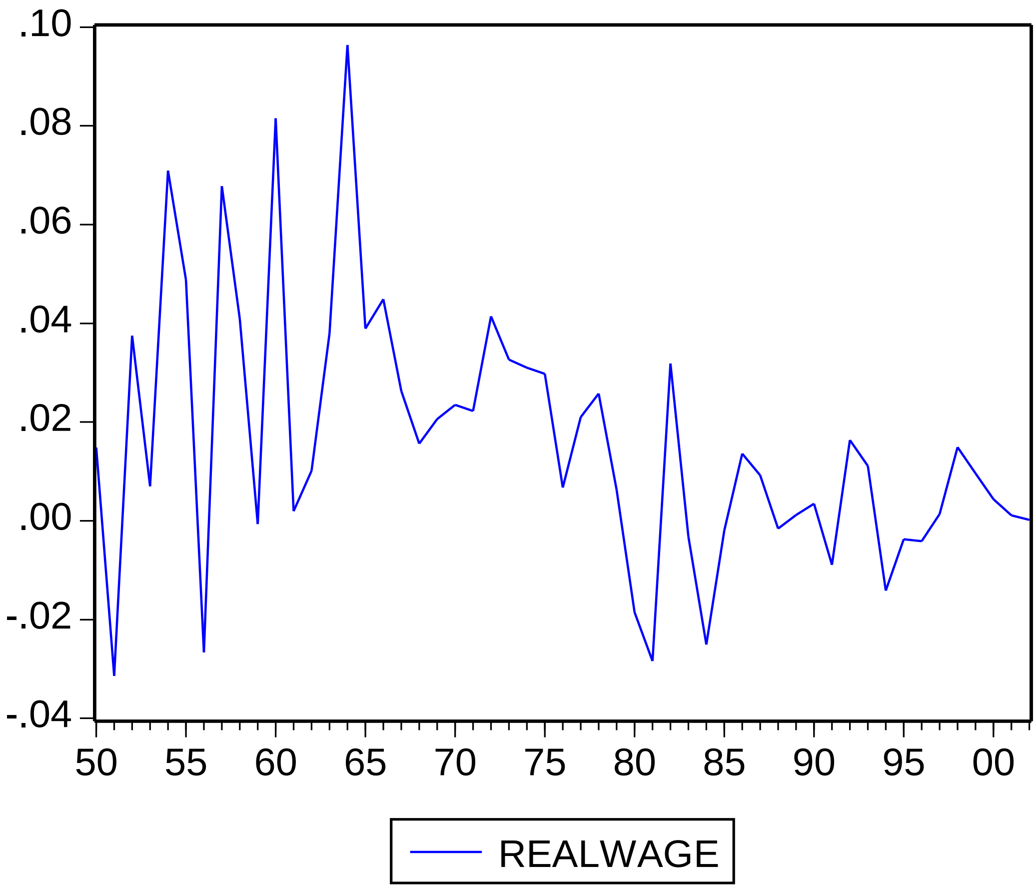

Table 2: VAR estimates for US data

This table shows estimates of a VAR on real wage growth, w_t, stock returns, R^S, interest rates, r, CPI inflation, π, log dividend yield, dp_t and the log wage-dividend ratio, y_t, using annual data for the US over the period 1947–2006. T-statistics are given in brackets, σ_e is the standard error of the equation.

	w_t	R_t^S	r_t	π_t	dp_t	y_t
w_{t-1}	0.147	–1.390	0.027	0.129	0.256	–0.623
	[1.64]	[2.04]	[0.29]	[2.26]	[0.51]	[2.67]
R_{t-1}^S	0.079	–0.280	0.012	0.027	–0.268	–0.025
	[4.43]	[2.06]	[0.62]	[2.36]	[2.68]	[0.53]
π_{t-1}	0.007	–2.273	0.632	0.347	1.378	–0.612
	[0.05]	[2.37]	[4.73]	[4.34]	[1.96]	[1.87]
r_{t-1}	–0.174	–0.849	0.004	0.680	–0.809	–0.322
	[1.64]	[1.04]	[0.04]	[10.04]	[1.36]	[1.16]
dp_{t-1}	0.006	0.216	0.007	0.004	0.913	–0.004
	[0.65]	[3.36]	[0.79]	[0.81]	[19.41]	[0.16]
y_{t-1}	–0.001	0.341	0.051	0.053	–0.223	0.923
	[0.03]	[1.61]	[1.72]	[2.99]	[1.43]	[12.72]
R^2	0.452	0.224	0.502	0.861	0.936	0.859

Table 3: Long run variances, correlations and risk premiums

This table shows the long-run standard deviation of the four variables, the exposure coefficients θ of the wage growth to the standardized innovations, the beta ($\beta_i = \theta_i/\sigma_i$). The Sharpe ratio's λ are assumed values, except for λ_w which is calculated as $\frac{1}{2}\gamma\theta_w$ for $\gamma = 5$. The final column μ shows the contribution of each factor to the risk premium on a long maturity wage-linked bond (in excess of the risk premium on a long-maturity nominal bond), calculated as $100\theta\lambda$ (i.e. expressed as percent per annum).

	$\sigma(k=1)$	$\sigma(k=100)$	w	R^S	π	r	θ	β	λ	$100\theta\lambda$
w	0.0204	**0.0449**	1	**0.707**	**–0.543**	**–0.728**	0.0240		–0.058	–0.14
R^S	0.1552	**0.1265**	0.039	1	**–0.184**	**–0.417**	0.0226	0.178	0.20	0.45
π	0.0217	**0.0543**	–0.348	–0.394	1	**0.825**	–0.0031	–0.058	–0.15	0.05
r	0.0129	**0.0835**	–0.160	–0.388	0.488	1	–0.0207	–0.248	–0.15	0.31