Variance Risk Dynamics, Variance Risk Premia, and Optimal Variance Swap Investments

Markus Leippold

Imperial College, Tanaka Business School
and
Centre of Quantitative Finance

joint work with Liuren Wu and Daniel Egloff

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Outline

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Motivation

- Variance is commonly used as a measure of the uncertainty in financial security returns, but variance is itself uncertain.
- When investing in an asset, the investor faces at least two sources of risk: (1) return, and (2) return variance.
- Variance-related derivative products grow rapidly in types and volume: Variance swaps, volatility swaps, options on realized variance, options on variance swaps.
- Questions:
  - How does variance vary?
  - How does the market price variance risk?
  - How should we use variance derivatives to expand our investment frontier?
- Answers to this question are fundamental for:
  - Asset allocation and risk management.
  - Derivatives pricing and hedging.
  - The general behavior of financial asset prices.

Variance Swap Rate

The contract:

- At inception, variance swap contract has zero value.
- At expiry, the long side receives the difference between the realized variance and a fixed rate (the variance swap rate):

$$P\&L = \left((\text{Ann. realized vol})^2 - (\text{ Variance swap rate})^2 \right) \times \text{Notional.}$$

Why variance swaps?

- The most actively traded and simplest variance derivative.
- Direct (linear) way for exposure to or hedging against variance risk.
- Unlike options, no interference with process specification of the underlying instrument.
- Allows the issuer a static replication via a portfolio of options.
- Data: S&P 500 variance swap data over five maturities from two months to two years, from January 1996 to March 2007.

What We Do

Variance risk dynamics and variance risk premia:

- Propose an affine class of models on variance risk dynamics.
- Estimate the variance risk dynamics and variance risk premia using both
  a) the time series and the term structure information of the variance swap rates
  b) the time series of realized variance.

Optimal variance swap investment:

- How does the presence of variance swaps alter an investor's asset allocation decision?
- How does variance swap investment influence the investor's welfare?

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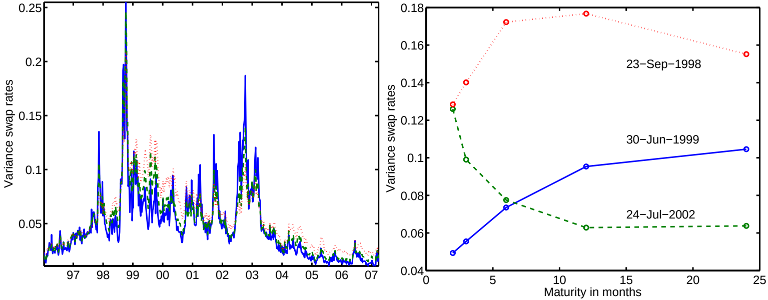
Variance Swap Rates: Time Series and Term Structure

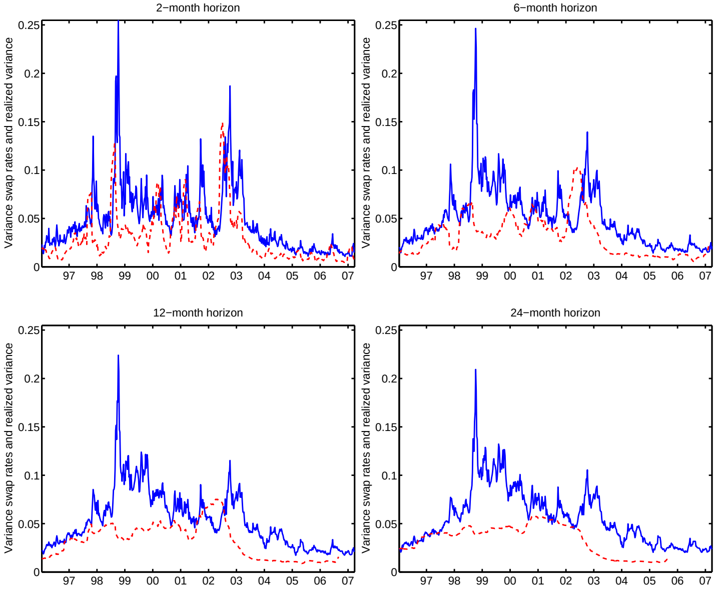


- Time-varying volatility;
- Various term structure shapes.

Data on Variance Swap Rates: Summary Statistics

Maturity	Mean	Std	Skew	Kurt	Auto
A. Variance swap rates, Maturity in months					
2	0.048	0.033	1.861	5.734	0.920
3	0.048	0.031	1.750	5.253	0.943
6	0.050	0.031	1.825	6.338	0.959
12	0.053	0.030	1.385	3.421	0.971
24	0.056	0.029	1.146	1.780	0.977
B. Annualized realized variance, Maturity in days					
7	0.031	0.044	4.900	33.500	0.382
30	0.032	0.031	2.524	7.937	0.925
60	0.030	0.026	1.958	4.188	0.970
90	0.031	0.025	1.651	2.924	0.982
120	0.031	0.024	1.498	2.480	0.989
150	0.031	0.022	1.315	1.805	0.992

Comparing Variance Swap Rate and Realized Variance



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The Framework

- $V_{t,T} \equiv \int_t^T v_s ds$ denotes the return variance during the period $\tau = T - t$.
- The $\mathbb{Q}$-dynamics of the instantaneous variance v_t is driven by a k-dimensional Markov process X, $v_t = v(X_t)$, with

$$dX_t = \mu(X_t)dt + \Sigma^X(X)dB_t^X + \left(qdN^X(\lambda(X_t)) - \bar{q}\lambda(X_t)dt\right),$$

- **Definition:**
  In **affine stochastic variance models**, the Laplace transform of the quadratic variation, $V_{t,T} = \int_t^T v_s ds$, under the risk-neutral measure $\mathbb{Q}$ is an exponential-affine function of the state vector X_t
- **Conditions:**
  Under $\mathbb{Q}$, the instantaneous variance rate v_t, the drift vector $\mu(X)$, the diffusion covariance matrix $\Sigma^X(X)\Sigma^X(X)^\top$, and the jump arrival rate $\lambda(X)$ of the Markov process X are all affine in X.

Model Specification and Design

Proposition: Affine Stochastic Volatility Models

- The variance swap rate at time t with maturity τ is just a linear combination of the state vector X_t at time t.
- The variance swap rate does not on the type and specification of the martingale component of the X_t vector.

Model Design:

1. One-factor model: $dv_t = \kappa_v (\theta - v_t)\, dt + \sigma_v \sqrt{v_t} dB_t^v$.
2. Two-factor model:

$$\begin{aligned} dv_t &= \kappa_v (m_t - v_t)\, dt + \sigma_v \sqrt{v_t} dB_t^v, \\ dm_t &= \kappa_m (\theta - m_t)\, dt + \sigma_m \sqrt{m_t} dB_t^m, \quad \mathbb{E}^{\mathbb{Q}} (dB_t^v, dB_t^m) = 0. \end{aligned}$$

- Variance Swap Rates:

$$VS_{t,\tau} = \phi_v(\tau) v_t + \phi_m(\tau) m_t + (1 - \phi_v(\tau) - \phi_m(\tau))\, \theta,$$

Variance Risk Premia

Market Price of Variance Risk:

$$\gamma(B_t^v) = \gamma^v \sqrt{v_t}, \qquad \gamma(B_t^m) = \gamma^m \sqrt{m_t}.$$

Term Structure Analysis:

Stylized Fact	Impact on Model Parameters		
Term structure slope:	1F	2F	Implication:
Mean up	$\theta_v^{\mathbb{P}} < \theta$ $\gamma^v < 0$	$\theta_v^{\mathbb{P}} < \theta_m^{\mathbb{P}} < \theta$ $\gamma^v < 0, \gamma^m < 0$	variance risk is priced
Std down	$\kappa_v > 0$	$\kappa_v, \kappa_m > 0$	stationary $\mathbb{Q}$ dynamics
Auto up	$\dagger$	$\kappa_v^{\mathbb{P}} > \kappa_m^{\mathbb{P}}$	two-factor model needed

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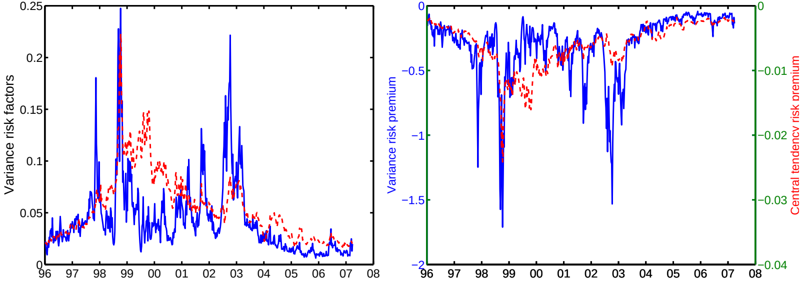
Estimation Results

X_t	κ	θ	σ	γ	$\kappa^{\mathbb{P}}$	$\theta^{\mathbb{P}}$
A: One-factor variance risk model						
v_t	0.1547 (13.81)	0.1220 (32.55)	0.2550 (47.47)	−17.0141 (43.06)	4.4929 (71.42)	0.0042 (21.52)
B: Two-factor variance risk model						
v_t	4.3730 (38.72)	–	0.4221 (44.10)	−16.3746 (37.70)	11.2851 (51.96)	0.0158 (3.71)
m_t	0.1022 (9.30)	0.0838 (24.31)	0.1581 (47.64)	−0.6844 (1.84)	0.2104 (3.40)	0.0407 (3.68)

- Estimated risk premia coefficients are large and negative!
- Reversion speed for v_t is much higher than for m_t: Half-life differences of ten weeks vs seven years.

Variance Risk Dynamics and Risk Premia



- Instantaneous variance rate v_t moves around central tendency factor m_t.
- Central tendency factor shows more persistence and less volatility.
- Same is true for the risk premia. Risk premia on m_t is very small in comparison.

Subsample Robustness

X_t	κ	θ	σ	γ	$\kappa^{\mathbb{P}}$	$\theta^{\mathbb{P}}$
Subsample period: 1996- 2001						
v_t	3.3945	–	0.4635	−16.2472	10.9250	0.0153
	(16.53)		(27.27)	(25.12)	(29.84)	(3.88)
m_t	0.1857	0.0715	0.2086	−0.4029	0.2697	0.0492
	(7.61)	(15.14)	(26.98)	(1.11)	(3.17)	(3.84)
Subsample period: 2001-2007						
v_t	5.3789	–	0.3476	−15.6418	10.8160	0.0239
	(33.83)		(27.86)	(19.44)	(33.91)	(0.46)
m_t	0.0010	4.2187	0.1058	−0.8444	0.0903	0.0480
	(0.06)	(0.06)	(24.42)	(0.46)	(0.46)	(0.46)

- Subsample estimates are very robust (for both one-factor and two factor model).
- Robustness gives hope for good out-of-sample performance of investment strategies.

Outline

The Allocation Problem

Question:

- How should we use variance swaps to expand our investment frontier?

Model setup for portfolio allocation:

- We specify the stock index dynamics as:

$$dS_t/S_t = \left(r + \gamma^S v_t\right) dt + \sqrt{v_t} dB_t^S, \quad dB_t^S dB_t^v = \rho dt.$$

- Investor can invest in the stock, the bond, and two(one) variance swaps.
- We equip investor with CRRA utility to max her end-of-period wealth.

Optimal Portfolio: One-Factor Model

- With accessibility of variance swaps:

$$
\begin{aligned}
w_t^S &= \frac{1}{\eta}\left(\gamma^S - \frac{\rho}{\sqrt{1-\rho^2}}\gamma^z\right), \\
w_t^{SV} &= \frac{1}{\eta\phi_v(\tau)}\left(\frac{\gamma^z}{\sqrt{1-\rho^2}\sigma_v} + h_v(u)\right),
\end{aligned}
$$

  with $h_v(u)$ the hedging demand and γ^z the pure variance risk premium ($\gamma^v = \rho\gamma^S + \sqrt{1-\rho^2}\gamma^z$).
- Without variance swap market: $w_t^S = \frac{1}{\eta}\left(\gamma^S + \rho\sigma_v h_v^M(u)\right)$,

Interpretation:

- Intertemporal hedging demand in w^S moves to w^{SV}.
- Pure variance risk premium γ^z becomes part of the myopic stock investment.
- If $\rho\gamma^z \gg 0$, then the investor might go short in the stock.

Optimal Portfolio: Two-Factor Model

- With accessibility of variance swaps (at least two!):

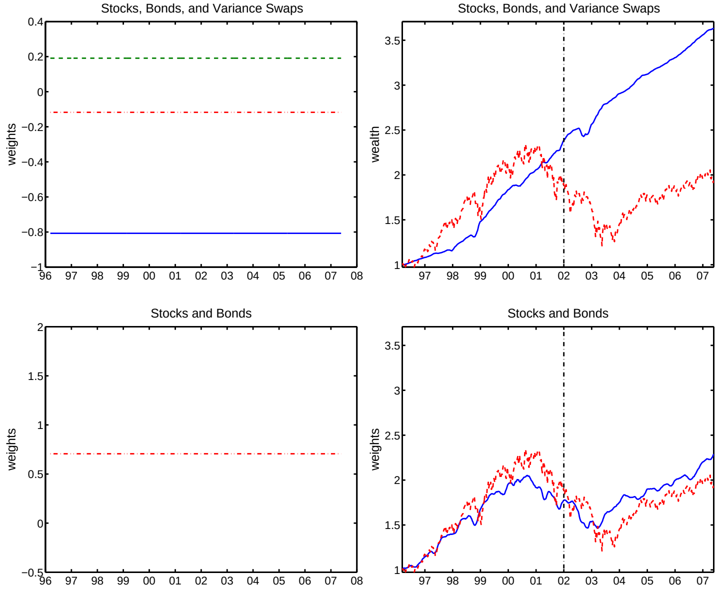
$$
\begin{aligned}
w_t^S &= \frac{1}{\eta}\left(\gamma^S - \frac{\rho}{\sqrt{1-\rho^2}}\gamma^z\right), \\
n_t^{\tau_1} &= \frac{1}{\eta D}\left[\left(\frac{\gamma^z}{\sigma_v\sqrt{1-\rho^2}} + h_v(u)\right)\phi_m(\tau_2) - \left(\frac{\gamma^m}{\sigma_m} + h_m(u)\right)\phi_v(\tau_2)\right], \\
n_t^{\tau_2} &= \frac{1}{\eta D}\left[-\left(\frac{\gamma^z}{\sigma_v\sqrt{1-\rho^2}} + h_v(u)\right)\phi_m(\tau_1) + \left(\frac{\gamma^m}{\sigma_m} + h_m(u)\right)\phi_v(\tau_1)\right].
\end{aligned}
$$

with intertemporal hedging demand $h_v(u), h_m(u)$.

Interpretation:

- If both variance risk premia are sufficiently negative, we short both contracts.
- Net position in variance swaps is always negative.
- Since γ^z is large compared to γ^m, we tend to go long the long-maturity swap.

Out-of-Sample Historical Performance



Summary Statistics

Strategy	Mean	Std	Skew	Kurt	Sharpe
A. Whole Period: 1996-2007					
$\mathcal{S}_1$	7.4303	4.9255	1.1945	7.0545	1.4984
$\mathcal{S}_2$	3.3106	10.7539	-0.1838	0.4997	0.3077
$\mathcal{S}_3$	7.9814	5.6856	1.0733	7.3502	1.3966
B. In-sample period: 1996-2001					
$\mathcal{S}_1$	9.0779	5.8340	1.3357	4.7407	1.6003
$\mathcal{S}_2$	6.1913	11.5308	-0.1158	0.2267	0.5396
$\mathcal{S}_3$	10.1087	6.6544	1.3312	5.1225	1.5699
C. Out-of-sample period: 2001-2007					
$\mathcal{S}_1$	5.9342	3.8114	-0.0381	3.8359	1.6522
$\mathcal{S}_2$	0.7085	10.0286	-0.4028	0.5058	0.0732
$\mathcal{S}_3$	6.0522	4.4910	-0.2057	4.8652	1.4083

Conclusion

How does variance vary?

- We need a two-factor stochastic variance model to explain the term structure variation of the variance swap rates.
- Variance rate factor is more transient than the central tendency factor under both probability measures.

How does the market price variance risk?

- Large negative variance risk premium on both variance risk factors.

How should we use variance derivatives?

- Variance swap is the simplest and most direct contract.
- Variance swaps remove the intertemporal hedging demand from the stock investment.
- Variance swap strategy does provide a smooth income stream.
- Stability of estimation allows for a very stable and high Sharpe ratio.