Estimating the Risk of Private Equity Funds: A New Methodology

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Abstract

We develop a new methodology to estimate the alpha and risk exposure of private equity funds. In contrast to existing work, our methodology mainly uses direct cash flow data and avoids the use of self-reported net asset values. Our GMM methodology is based on pricing restrictions for the cross-section of funds. We apply our methodology to a unique dataset comprising 23,296 cash-flows from 941 private equity funds between 1980 and 2003. We find a high market beta, especially for venture capital funds and report evidence that returns resemble those of long positions on call options. We also find evidence that venture capital funds loads positively on SMB and negatively on HML while buyout funds loads negatively on SMB and positively on HML. Only buyout funds are exposed to systematic liquidity risk.

1 Introduction

Private equity funds are financial intermediaries that invest mainly in venture capital and leveraged buyouts. Their investors, such as Endowments and Pension Funds, commit capital to these funds instead of investing directly is these assets. In 2005, a record high amount of $200 billions were invested in private equity funds. The main objective of this paper is to measure the risk faced by investors on their capital allocated to private equity funds.

Currently, the literature offers both estimates of the risk of venture capital investments gross of fees (e.g. Cochrane, 2005a) and estimates of the risk of private equity funds net of fees (e.g. Jones and Rhodes-Kropf, 2004). The first set of studies measure the risk faced by the fund managers on their venture capital investments. This risk is related but different from the risk faced by fund investors (Limited Partners, LPs). The reason is that the cash-flow faced by LPs comprise, but are not limited to, the investments and divestments of the venture capital investments. Funds, even those specialized in venture capital, make other type of investments and, importantly, charge fees in a non-proportional fashion. Hence, to evaluate the risk faced by an LP, using data on the net cash-flow to/from LPs should improve risk estimates.

This is what the second set of studies do and the main study in this set is that of Jones and Rhodes-Kropf (2004). They assume that the accounting values reported quarterly by private equity funds are an unbiased, though stale, estimates of market values. They obtain alpha and betas of portfolio of funds by regressing portfolio returns[1] on both contemporaneous and lagged risk factors. Their contribution is an important and substantial step towards estimating private equity fund risk profiles. However, we find via a Monte Carlo simulation that such an approach generates large and systematic errors for estimates of both risk and abnormal performance. We also derive the bias in closed form and show that it cannot be trivially corrected in a standard OLS setup.

To solve this problem, we estimate risk loadings with a method of moments by using a unique dataset comprising 23,296 cash-flows from 941 private equity funds between 1980 and 2003. Formally, the idea of our new methodology is to specify the functional form of the stochastic discount factor (SDF) m_{0t} (from time 0 to t) and to find parameters in the SDF such that $V_0 = \sum_t E(m_{0t}x_t)$, where V_0 is the value of the fund (typically zero as most funds in our sample are fully liquidated) and x_t is the cash flow at time t. Setting the problem in this form is natural and intuitively appealing (see Cochrane, 2005b).

In practice, we observe a stream of cash flows (dividends and investments) and a final value. For each fund, we then have 1 equation in n unknowns, n being the number of free parameters in the stochastic discount factor. Using a cross-section of $N > n$ funds, we obtain an overidentified system of equations and can apply a Generalized Method of Moment approach to find the n free parameters. In addition, the method of moments allows us to leave the distribution of the errors unconstrained. This is an appealing feature in private equity as the return distribution is likely to differ from (log-)normality.

We find that the CAPM-beta of funds is 1.05. In addition and importantly, we find that the CAPM-beta decreases with positive stock-market performance. This indicates that funds offer a similar risk profile as call options (See Coval and Shumway, 2001). This finding is also similar to what Agarwal and Naik (2004) document for Hedge Funds. Taking into account this non-linearity is key and as far as we know, it has been missing in the literature. Incorporating this effect leads to a time-varying CAPM-beta, reaching levels between 0.1 and 1.7.

Moreover, we find that funds are exposed to liquidity risk as measured by Pástor and Stambaugh (2003). Funds also offer positive loadings on SMB and

[1] R_t is obtained by $\frac{RV_{t+1}+D_t-T_t}{RV_t}$, where RV is accounting values of the fund, D is the dividend and T is the investment.

negative loadings on HML. The results for the Fama-French factors (SMB and HML) are, however, very different for funds that focus on buyout and for funds that focus on venture capital. As expected, venture capital funds look like small growth stocks (positive loading on SMB and negative on HML) whereas buyout funds look like large value stocks. It is also worth noting that venture capital appears to have betas that are substantially higher than those of buyout funds. Buyout funds offer a CAPM-beta of 0.66. This is similar, albeit higher, to what Jones and Rhodes-Kropf (2004) report and is quite puzzling given the high amount of leverage used by buyout funds and the fact that their investments are similar to publicly traded companies. Venture capital funds offer a CAPM-beta of 1.23 in the CAPM specification.

Finally, results are robust to the weight assigned to moment conditions (equally weighted versus size weighted) and sample selection (funds raised in 1980-1993 versus 1980-1996).

Knowing the risk profile of different type of private equity funds enables investors to improve their asset allocation to the private equity asset class and across private equity funds. Our methodology also permit to better mark to market private equity investments and to evaluate the relative performance of private equity funds compared to their peers and compared to other benchmarks such as public equity or bonds.

The rest of the paper proceeds as follows. Section I presents the data; Section II is dedicated to the methodology; Section III shows the results from the Monte Carlo simulation and discusses the bias in the traditional approach; Section IV presents the results and Section V briefly concludes.

2 Data

2.1 Institutional details

The private equity funds in our study are organized as limited partnership and have a finite life (10 years extensible to 14 years). This structure is by far the most common in this industry. Investors, called Limited Partners (LPs), are principally institutional investors. LPs commit a certain amount of capital to private equity funds, which are run by General Partners (GPs). In general, when a GP identifies an investment opportunity, it "calls" money from its LPs up to the amount committed (undiscounted), and at any point in time until the liquidation of the fund. Such "calls" are called drawdowns or takedowns. When an investment is liquidated, the GP distributes the proceeds to its LPs either in kind or in cash. The timing of these cash flows is typically unknown ex ante. Compensation from LPs to GPs consists of (i) a management fee based on either the amount invested (undiscounted), or the capital committed, or a combination of the two and (ii) a fraction of profits called carried interest (with profit being defined differently across funds).

2.2 Data source

Data on private equity funds are from Thomson Venture Economics (TVE). TVE records the amount and date of all the cash flows as well as the aggregate quarterly book value of all unrealized investments (residual values) for each fund from 1980 to 2003. Cash flows are net of fees as they include all fee payments to GPs, including carried interest.

Venture Economics offer the most comprehensive source of financial performance of both US and European private equity funds and has been used in previous studies (e.g., Kaplan and Schoar, 2005). It covers an estimated 88% of venture funds and 50% of buyout funds in terms of capital committed. TVE builds and maintains this dataset based on voluntarily reported information about cash flows between GPs and LPs in Private Equity funds. TVE obtains and crosschecks information from both GPs and LPs, which increases the reliability of this dataset. Finally, the aggregate residual values of unrealized investments (i.e., non-exited investments) are obtained by TVE from audited financial reports of the partnership.

Data on Treasury bill rates, stock performance and liquidity factors are from WRDS. Data on corporate bond yields are from the Federal Reserve Bank of Saint Louis.

2.3 Sample selection

We select our main sample is the same fashion as Kaplan and Schoar (2005). A fund is included in our database if it is raised between 1980 and 1996 and is either officially liquidated as of December 2003 or has not reported any cash flow during the last 6 quarters (July 2002 to December 2003). Such funds are called quasi-liquidated as they are close to their liquidation time.

Descriptive statistics are reported in Table 1. BO funds are larger than VC funds. VC funds are mostly raised in the United States. VC funds have more dividend payouts and also higher frequency than BO funds. In total, we have 23,296 cash flows in the quasi-liquidated sample with 15,731 are VC funds and 7,565 are BO funds. The cash flows come from 941 funds with 673 are VC funds and 268 are BO funds.

As mentioned above PE funds have a natural life of 10 years. A natural sample to select is therefore funds raised from 1980 to 1993. Such a sample is used in the robustness section.

3 Methodology

3.1 Example

Private Equity (PE) funds are not publicly traded. Hence, one cannot estimate the alpha and factor-exposures (the betas) directly by the traditional regression method. Below we propose a methodology to infer these parameters using panel data of the cash flows of private equity funds.

Let us illustrate our methodology with a simple case. Let us assume that a fund is priced according to the Capital Asset Pricing Model (CAPM) and we observe the following cash flow stream (these are cash amounts to/from LPs.): a takedown of 100 occurs at time 1, a takedown of 200 occurs at time 2, a distribution of 180 occurs at time 3 and of 200 occurs at time 4. The fund is then reported as liquidated. For simplicity, assume the market return is 10% and risk-free rate is 5% for each period. If the CAPM is the correct model to describe fund's returns and all the risk is systematic (i.e., there are no idiosyncratic shocks) then we have the following relationship.

$$\begin{aligned} V(4) &= 100[1+5\%+\beta(10\%-5\%)]^3 \\ &\quad +200[1+5\%+\beta(10\%-5\%)]^2 \\ &\quad -180[1+5\%+\beta(10\%-5\%)]-200 \\ &= 0 \end{aligned}$$

That is, the investment of 100 grows at the CAPM rate until liquidation, hence for 3 periods, the investment of 200 grows at the CAPM rate until liquidation, hence for 2 periods. There is one intermediate dividend that decreases the market value of the fund at date 3. Finally the liquidation value is 200. This equation should hold. With our assumptions, there is only 1 unknown and the solution is $\beta = 1.71$. In reality, of course, private equity funds are expected to exhibit considerable idiosyncratic risk. Hence we need to take expectations of the compounded fund value and use a cross-section of funds to assess which beta is most appropriate. In this example, if the fund was publicly traded we would observe 4 market values. As it is not, we observe only 1 (the final one). However, since we have several funds (about 1000), we can still estimate the risk parameters.

3.2 Formal derivation

We define the value of the project j of the fund i as

$$\begin{aligned} V_{ij}(\alpha,\beta) &= (D_{ij}-T_{ij})\prod_{t=t_{ij}}^{t=l_{ij}}(1+r_t+\alpha+\beta r_{m,t}) \\ &= D_{ij}\prod_{t=t_{ij}}^{t=l_{ij}}(1+r_t+\alpha+\beta r_{m,t}) - T_{ij}\prod_{t=t_{ij}}^{t=l_{ij}}(1+r_t+\alpha+\beta r_{m,t}) \\ &: = V^{D_{ij}} - V^{T_{ij}} \end{aligned}$$

where $r_{m,t}$ is excess market return, l_{ij} is the liquidation date of the project, t_{ij} is the date of the first investment.

We assume the following error structure

$$\begin{aligned} V^{T_{ij}}(\alpha,\beta) &= E[V^{T_{ij}}(\alpha,\beta)] + \eta_{ij} \\ V^{D_{ij}}(\alpha,\beta) &= E[V^{D_{ij}}(\alpha,\beta)] + \upsilon_{ij} \end{aligned}$$

where η_{ij} and υ_{ij} are zero-expectation error terms. Thus, our (mis)pricing model is similar to Cochrane (2005a), who also assumes an additive mispricing parameter in his setup. It is also useful to consider asymptotics with respect to the number of projects per fund[2], where we will assume that projects do not overlap in time. In this way, adding more projects increases the life of the fund.

The key step to the parameter identification is the moment restriction

$$\begin{aligned} & E[V_{ij}(\alpha,\beta)] = 0 \\ \Rightarrow \quad & E[V^{D_{ij}}(\alpha,\beta)] = E[V^{T_{ij}}(\alpha,\beta)], \qquad i = 1,...,N \ ; \ j = 1,..,n_i \end{aligned} \tag{1}$$

where N is the number of funds in our sample, and n_i the number of projects of fund i. That is to say, if the pricing model is correct, the expected value of each project equals zero[3]. Since we only have the data at the fund level rather than at the project level, we sum over all the investments and dividends in a fund and obtain the sample moments:

$$\begin{aligned} \overline{V}^{T_i}(\alpha,\beta) &= \frac{\sum_{j=1}^{n_i} V^{T_{ij}}(\alpha,\beta)}{n_i} \\ &= \frac{V^{T_{ij}}(\alpha,\beta)}{n_i} \qquad\qquad i = 1,...,N \end{aligned}$$

$$\begin{aligned} \overline{V}^{D_i}(\alpha,\beta) &= \frac{\sum_{j=1}^{n_i} V^{D_{ij}}(\alpha,\beta)}{n_i} \\ &= \frac{V^{D_{ij}}(\alpha,\beta)}{n_i} \qquad\qquad i = 1,...,N \end{aligned}$$

[2]This could either be really in terms of the number of projects, or alternatively in terms of dollars invested, thus assuming unit dollar projects.

[3]Our pricing model includes a "mispricing" parameter α.

Next, we take logs of the sample moment conditions[4].

$$\begin{aligned} & \log(\overline{V}^{T_i}(\alpha,\beta)) = \log(\overline{V}^{D_i}(\alpha,\beta)) \\ \Rightarrow\quad & \log(\frac{V^{T_i}(\alpha,\beta))}{n_i}) = \log(\frac{V^{D_i}(\alpha,\beta)}{n_i}) \\ \Rightarrow\quad & \log(V^{T_i}(\alpha,\beta))) = \log(V^{D_i}(\alpha,\beta)) \end{aligned}$$

The criterion function to be minimized becomes

$$\ln V^{T_i}(\alpha,\beta)) - \ln V^{D_i}(\alpha,\beta)$$

The estimation can be done by applying the GMM to the equation (1). The first-step GMM consists in solving

$$\min_{\alpha,\beta}\sum_{i=1}^{N}[\ln V^{T_i}(\alpha,\beta) - \ln V^{D_i}(\alpha,\beta)]^2 \tag{2}$$

Assuming that idiosyncratic shocks of projects within and across funds are independent[5], we can apply a central limit theorem to obtain consistency of the estimator and the asymptotic distribution. Consistency here means that

$$-\frac{V^{T_i}(\alpha,\beta))}{n_i}+\frac{V^{D_i}(\alpha,\beta)}{n_i} \rightarrow^P -E[V^{T_{ij}}(\alpha,\beta))]+E[V^{D_{ij}}(\alpha,\beta)] = 0, \quad as \quad n_i \rightarrow \infty$$

and in the log form

$$-\ln V^{T_i}(\alpha,\beta))+\ln V^{D_i}(\alpha,\beta) \rightarrow^P -\ln E[V^{T_{ij}}(\alpha,\beta))]+\ln E[V^{D_{ij}}(\alpha,\beta)] = 0, \quad as \quad n_i \rightarrow \infty$$

Standard errors, however, cannot be obtained by the standard GMM approach. Without the returns of projects in each fund, we can not estimate the variance of the project value within a fund. But if we assume that idiosyncratic shocks of projects are independent within and across fund, we can obtain the standard errors of alpha and beta by bootstrapping.

We first draw the same amount of funds from our quasi-liquidated sample with replacement, and then re-estimate the alpha and beta. Repeating the process 1,000 times, we have the distribution of the alpha and beta. Although we draw a batch of projects (all projects within a fund) rather than only one

[4] This is done to improve small-sample properties. Since we use compounded return, it is very likely that the distribution of the present value of the funds are skewed to the right as in the lognormal distribution. Our simulation study shows that taking log helps to improve the small-sample properties. Cochrane (2005a) also assumes a lognormal distribution for the growth of VC fund value.

[5] This assumption is satisfied naturally if we assume projects to be active in different time periods. Meanwhile, after controlling the factor returns, any systematic patter should be incorporated if the pricing model is correct.

project each time, the asymptotic property of standard error still holds as long as all idiosyncratic shocks of projects are independent[6].

Our specification can accommodate time-variation in the beta. In this paper, we focus on time-variation in the beta of the type observed for options. Coval and Shumway (2001) follow Black and Scholes (1973) and use CAPM to derive the beta of a call option is

$$\beta_c = \frac{S}{C}\mathcal{N}(d_1)\beta_s$$

They show that the Black-Sholes betas of the call options increase with the difference between option's strike price and underlying price. That is to say, given a strike price, the option beta decreases as underlying price increases. The more the option is in the money, the lower the beta of the call option.

If private equity funds have the feature of a call option as Cochrane (2005a) suggests, the beta of private equity fund should vary with the moneyness of the option. In this case, the fund's β can be modeled as a parametric function of observables, and the parameters in this specification can be estimated along with α in the same way as above. For example, to mimic the beta of call options, we model the β of PE funds for period t as follows

$$\beta_t = \beta_0 + \beta_1 \log \frac{S_t}{S_0} \tag{3}$$

where S_t is the value of the stock market index at period t (so $log\frac{S_t}{S_0}$ is just the multi-period return), and S_0 denoting the value of the stock market index at the beginning the vintage year of funds (then S_0 will depend on individual fund). This incorporates that the beta of a call option changes as the underlying value changes. More specifically, the beta decreases as the investment becomes more "in-the-money", i.e. as the stock market value increases. Since projects of PE funds take few years to payoff, they are long-term call options. Our setup is in line with this feature. Thus, three parameters would need to be estimated, α, β_0, and β_1.

[6] To show that drawing a group of variables rather one variable each time has the same result, consider the example of variance of the sample mean. If we group $x_1..x_n$ random variables that are *i.i.d.* with a mean $\overline{x}$ into m groups, each group has l variables and an average value $\widetilde{x}_i$. The variance of the sample mean follows:

$$Var(\overline{\widetilde{x}}) = \frac{Var(\widetilde{x}_i)}{m} = \frac{Var(x_i)}{ml} = \frac{Var(x_i)}{n} = Var(\overline{x})$$

where $\overline{\widetilde{x}} = \sum_i^l \widetilde{x}_i$

Hence, bootstrapping on the individual variable level or on group level should give the same variance. Since we assume that the idiosyncraic shocks of PE funds are *i.i.d.*, the the property should hold.

4 Simulation

4.1 Traditional approach

The traditional approach consists in assuming that accounting values are unbiased assessments of the fund market values and that they are infrequently updated. In this case, researchers use the time-series of accounting values, correcting for this "stale pricing" problem. The traditional approach[7] , invoking Dimson (1979), consists in first aggregating all the funds to obtain one time-series of returns and, second, regressing the time-series of returns on to contemporaneous and lagged risk factors. The accounting value of PE funds is the Residual Value (RV), which is an accounting report of the of all non-exited investments. The one period return follows

$$R_t = \frac{(AggRV_{t+1} + AggDiv_t - AggInv_t)}{AggRV_t} - 1$$

where $AggRV_t$ is the aggregate RV, $AggDiv_t$ is the aggregate dividends and $AggInv_t$ is the aggregate investments. Betas are then estimated via an OLS regression:

$$r_t = \alpha + \beta_0 r_{m,t} + \beta_1 r_{m,t-1} + ... + \beta_4 r_{m,t-4} + \varepsilon_t$$

where all returns are excess returns over the T-bill rate.

The idea is that betas can be consistently estimated by simply summing up the betas on current and lagged factor returns (the aggregate beta in Dimson's paper). In a simulated private equity fund economy, however, we show that such a correction leads to biased results.

4.2 Monte Carlo experiment

To evaluate and compare the performance of our GMM methodology with the traditional approach, we run a Monte Carlo experiment.

We assume that the market value of private equity fund follows the CAPM up to a constant and idiosyncratic shocks. Researchers observe the accounting values, investments and dividends of 300 funds over 25 years at a quarterly frequency. For the first 15 years, 20 funds invest all their capital, which is normalized to 1, at the beginning of the year. Each quarter, with certain probability, funds distribute a dividend which is a fraction of the market value of the fund, until the market value of the fund reaches a lower threshold. When this event occurs, the fund is liquidated. This setting is similar to the data set in the hand of researchers. Then the fund value, V^i_{pe}, evolves as

$$V^i_{pe,t} = V^i_{pe,t-1}(1 + \alpha + r_f + \beta r_{m,t-1} + \varepsilon^i_{t-1})$$

And if the dividend is paid, it follows

[7] See Woodward (2004) and Jones and Rhodes-Kropf (2004).

$$d_t^i = V_{pe,t-1}^i(1 + \alpha + r_f + \beta r_{m,t-1} + \varepsilon_{t-1}^i)\pi$$

where π is the dividend payout ratio.

Accounting values are simulated as follows. The revelation state, which means the RV is observed, occurs with probability θ. The revelation state does not occur with probability $(1-\theta)$, in which case the accounting value equals the accounting value of the previous quarter. In the revelation state, the accounting value equals the market value. The probability for the revelation state is set to 1/8, to reflect the claim in Woodward (2004) that investments are evaluated every 8 quarters on average. An assumption that implies that every 8 quarters, the truth is revealed with minor error. This is likely an optimistic assumption that works in favor of the traditional approach approach. In this simulated economy, we assume that accounting values are unbiased. If, in practice, they are biased or noisier than we have assumed, then results of the traditional approach will be noisier than what is found in these simulations. As we do not use accounting values, our GMM methodology is more robust to assumptions regarding their accuracy.

We assume a risk-free rate of 1% per quarter, a beta of 1.5, and alpha of 1% per quarter, a stock market portfolio volatility of 10% per quarter. We also calibrate the dividend payout ratio and the frequency of dividend payout as 1/5 and 1/4 respectively. This means overall funds pay 20% of their market value to the investors per year. Finally, the liquidation threshold is set to be 10%, which means if the fund value is below 0.1, it will be liquidated and the residual value will be treated as the last dividend payout. The parameters are calibrated to match the characteristics of the dataset we have. Both the size of the dataset and the resulting average time duration of the funds are similar to what is faced in practice (i.e. in the dataset used by Jones and Rhodes-Kropf (2004) and us). We also assume the idiosyncratic shock for an individual fund is normal i.i.d. and has 13%[8] volatility per quarter.

If we assume that there is no stale pricing problem, that is to say the market values of the PE funds are revealed every quarter, the traditional approach does a good job in estimating alpha and beta. This is because they aggregate the value of individual fund before running the regression. By doing so, the idiosyncratic shocks are averaged out and adding more lag terms only decreases the precision of the estimation. However, perfect revelation is an unrealistic assumption. Table 2 - Panel B reports that, if the market values of the PE funds are on average only revealed every 2 years and we first set idiosyncratic shocks zero for comparison, results are striking. Even if we include 8 quarters of lagged return on the market portfolio, the alpha and beta of the PE funds are found to be significantly biased downward in the traditional approach. The mean and median of the aggregate beta with 8 lags are 1.17 and 1.16 respectively, which is far away from the true value 1.5. With our methodology, in contrast, we find that the average estimate of beta remains by and large 1.5. This shows

[8] This number is backed out by assuming PE funds have 40% volatility per annum.

that as long as the market value of PE fund follows the CAPM, our method can identify the alpha and beta cross-sectionally without being severely affected by the nonsynchronous problem. This provides a good solution to the stale pricing in PE funds.

Now we allow idiosyncratic shocks for the PE funds. Table 2 - Panel C shows that traditional approach is not affected by the idiosyncratic shocks as mentioned above, but remains biased. Idiosyncratic shocks add more noise to the GMM method. This is because the idiosyncratic shocks cannot be averaged out fully by aggregation in the nonlinear framework. This makes it interesting to see the large sample property of the two models in terms of periods, vintage years and the fund numbers. By increasing the sample size, we find that our GMM methodology is asymptotically consistent while the traditional approach remains biased and is thus inconsistent. Table 3 - Panel B shows that our method identifies the true parameter if there are no idiosyncratic shocks. But the aggregate beta with 8 lags is still biased downward with a mean and median estimate for the beta of 1.25. Adding idiosyncratic shocks to the model in Table 3 - Panel C and the precision of the estimate is improved compared with Table 2 - Panel C by our method.

After observing a bias in simulations, we modelize theoretically the problem in order to derive the bias in closed-form. To begin, let us assume that an observed price $\widehat{P}_{i,t}$ may either represent the true price $P_{i,t}$ or the observed price at the previous period $\widehat{P}_{i,t-1}$ for fund i. If a stock suffers from thin trading problem or a private equity fund does not update its NAVs per quarter, the price revelation process should be:

$$E(\widehat{P}_{i,t}) = \theta P_{i,t} + (1-\theta)\widehat{P}_{i,t-1} \tag{4}$$

The expectation of the observed price at t then is simply the weighted average depending on the revelation frequency θ (in terms of private equity funds, the NAVs update frequency). This specification differs slightly from Dimson (1979). Appendix 1 shows that Dimson's specification also leads systematic biases when estimating β. We can derive the aggregate beta as Dimson does[9]:

$$[\theta\frac{P_{i,t-1}}{\widehat{P}_{i,t-1}} + (1-\theta)\theta\frac{P_{i,t-2}}{\widehat{P}_{i,t-1}} + (1-\theta)^2\theta\frac{P_{i,t-3}}{\widehat{P}_{i,t-1}} + ...]\beta = \sum_{j=0}^{t}\widehat{\beta}_{-j}$$

We can see that even if t goes to infinity, the aggregate beta will not converge to the true beta unless $\frac{P_i}{\widehat{P}_i}$ on average is close to one. But this is not the case if prices have a positive drift. In the context of private equity, the bias is much more severe because the NAVs are reported on quarterly basis. It is unrealistic to assume quarterly NAVs do not grow over time. Therefore, $\frac{P_{i,t-j}}{\widehat{P}_{i,t-1}}$ will be larger than one for the first few j periods depending on the size of growth rate and the revelation frequency; $\frac{P_{i,t-j}}{\widehat{P}_{i,t-1}}$ will be smaller than one for the rest

[9] see Appendix 1 for the discussion of Dimson's model and Apendix 2 for the derivation.

periods. However, the influence of $\frac{P_{i,t-j}}{\widehat{P}_{i,t-1}}$ becomes negligible after certain periods because of the diminishing $(1-\theta)^j$ terms. We would expect the $\frac{P_{i,t-j}}{\widehat{P}_{i,t-1}} > 1$ part dominates the direction of bias. And this causes the estimated beta upward biased.

On the other hand, even if we assume that $\frac{P_i}{\widehat{P}_i}$ is very close to 1, including only limited lagged terms (such as 4 lagged terms in the JRK's paper) gives a downward biased result. This is because

$$\sum_{j=0}^{\infty} \theta(1-\theta)^j = \theta \frac{1}{1-(1-\theta)} = 1$$

To summarize, there are two opposite forces that drive the estimate away from the true value when the Dimson's aggregate beta is applied to the stale pricing problem of private equity funds. To see the magnitude of these two opposite effects, we rerun a simpler simulation. The true price follows a CAPM with alpha equals to zero and beta equals to one. The observed price follows equation (4). and The result is in Table 4. We can see that when price has no positive drift, the aggregate converges to the true one after adding 50 lags. Including only 8 or even 20 lags still leads to a downward biased result, 0.7 and 0.94 respectively. On the other hand, positive drift leads to a upward biased result in adding 20 and 50 lags cases. But in less lags cases, the aggregate betas are still downward biased.

In reality, it reduces the sample size too much if 20 lagged market returns are added to estimate the aggregate beta since private equity funds normally last for 40 quarters. Moreover, the growth rate of the stock market is not zero either. Our simulation result shows that biases due to first effect dominates. But the direction of biases for real data is ambiguous. In contrast, we have shown that our GMM gives an unbiased and consistent estimation by simulation when the stale pricing problem exists. In other words, the unbiasedness and consistency does not hinge on the assumptions of revelation frequency and growth rate of the market return. Next, the GMM model is applied after a brief description of the data in the next section.

5 Empirical Results

A. Risk profile of private equity funds

We start by estimating the risk profile for the 941 funds in our main sample. Results are reported in Table 5 - Panel A. The CAPM specification indicates a beta of 1.05 and a statistically significantly negative alpha of about -10% per year. Such a result mirrors the findings of Phalippou and Gottschalg (2006) of low private equity fund performance. When the two Fama-French factors are added, alpha increases because funds overall are similar to small growth stocks, which have low performance over this time period. The CAPM-beta stays stable at about 1.08. Results are very similar when the liquidity risk factor is added as the loading is not statistically significant.

The fourth specification is the one that allows for time-varying CAPM-beta as described in equation (3) above. Interestingly, we find that Beta1, the time-varying component of beta, is negative and significant, which shows that the CAPM-beta decreases with stock-market performance. This indicates that funds offer a similar risk profile as call options. Indeed, the beta of a call option decreases with its moneyness. As the price of the underlying rises, the return of a call option becomes less sensitive to the underlying asset (the delta converges towards one). As we take the stock market portfolio (S&P 500) as the underlying asset, this shows that holding private equity funds is similar to hold a long-term call option on the market portfolio. This conjecture has often been formulated in the literature (see Cochrane, 2005a) but, to our knowledge, never been empirically tested. Taking into account this non-linearity is therefore key to properly estimate the CAPM-beta. If the return resembles that of a call option, the CAPM beta varies considerably over time. Averaged across all funds, the minimum beta over time is 0.1, while the maximum beta equals 1.7. Results are similar after controlling for the two factors of Fama-French and liquidity risk. This non-linearity therefore plays an important role in generating time variation in the beta.

As BO and VC funds operate in different market segments, their risk profiles might differ. We therefore separate the funds that focus on VC from those that focus on BO. The advantage is that we have a more homogeneous group of funds, which should help obtaining more accurate estimates but this comes at the expense of less observations and thus increased potential error.

Results for VC are reported in Table 5 - Panel B and results for BO are reported in Table 5 - Panel C. VCs are found to have a higher beta (1.23) than BO funds (0.66). After taking the call option feature into account, both betas increase with similar magnitude for VC (from 1.23 to 1.89) and for BO (from 0.66 to 1.04). Beta1 is larger for VC showing that the call option feature is more pronounced for VC than for BOs.

An important difference exists between VC and BO regarding their exposure to the other three factors. As expected, venture capital funds look like small growth stocks (positive loading on SMB and negative on HML) whereas buyout funds look like large value stocks.

B. Comparison with literature's estimates

Our estimate of beta is similar to what has been documented in the literature. The beta on VCs without accounting for time-varying beta is 1.23, which is close to the 1.1 figure advanced by Ljungqvist and Richardson (2003). Our 1.9 beta estimate for VC after accounting for time-varying beta is similar to the 1.8 found by Jones and Rhodes-Kropf (2004).

Beta for BO without time-variation adjustment is also close to the 0.66 estimate of Jones and Rhodes-Kropf (2004) but we show that beta increases dramatically after accounting for time variation.

Cochrane (2005a) estimate for beta for arithmetic returns is 2.0 with standard error 0.6. Taking log returns to trim the outliers produces a lower beta of 0.4 with standard error 0.1.

It might seem puzzling that betas seem always to be much higher for VC than for BO. Especially given the high amount of leverage used by buyout funds and the fact that their investments are similar to publicly traded companies.

C. Robustness Check

First, we give a higher weight to the bigger funds. The moment conditions are weighted according to fund's log size at December 2003. The results are in the Table 6-Panel A, B, C for all funds, VC funds and BO funds respectively. The betas are qualitatively similar. The call option feature is significant for all models except for the BO funds only if we do not control other risk factors. Second, we use a subset of the quasi-liquidated sample which consists funds raised between 1980 to 1993. Table 7 shows that the betas are qualitative similar but less significant probably due to a smaller sample size. Our results are therefore robust to the weight assigned to moment conditions (equally weighted versus size weighted) and sample selection (funds raised in 1980-1993 versus 1980-1996).

6 Conclusion

We find that the CAPM-beta decreases with stock-market performance. This indicates that funds offer a similar risk profile as call options. This finding is also similar to what is documented for Hedge Funds. Taking into account this non-linearity is key and as far as we know, it has been missing in the literature. When such a feature is controlled for, the CAPM-beta goes up significantly.

Moreover, we find that funds are exposed to liquidity risk as measured by Pástor and Stambaugh (2003). Funds also offer positive loadings on SMB and negative loadings on HML. The results for the Fama-French factors (SMB and HML) are, however, very different for funds that focus on buyout and for funds that focus on venture capital. As expected, venture capital funds look like small growth stocks (positive loading on SMB and negative on HML) whereas buyout funds look like large value stocks. It is also worth noting that venture capital appears to have betas that are substantially higher than those of buyout funds. Finally, results are robust to the weight assigned to moment conditions (equally weighted versus size weighted) and sample selection (funds raised in 1980-1993 versus 1980-1996).

Knowing the risk profile of different type of private equity funds enables investors to improve their asset allocation to the private equity asset class and across private equity funds. Our methodology also permit to better mark to market private equity investments and to evaluate the relative performance of private equity funds compared to their peers and compared to other benchmarks such as public equity or bonds.

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Appendix 1: Dimson's aggregate beta

Dimson (1979) argues that since securities are traded intermittently, an observed price $\widehat{P}_t$ may represent a transaction price P_t, in the same period t or a price P_{t-i} established in the last trade which occurred in period $t-i$ $(i>0)$. Observed prices therefore have an expected value which is a weighted average of a sequence of true prices, where the latter are the transaction prices which would arise if trading were continuous,

$$E(\widehat{P}_t) = \sum_{i=0}^{n} \theta_i P_{t-i}$$

and

$$E(\triangle \widehat{P}_t) = \sum_{i=0}^{n} \theta_i \triangle P_{t-i}$$

The continuously compounded return $\widehat{R}_t$, based on observed prices can be obtained after a log-transformation,

$$\begin{aligned} E(\widehat{R}_t) &\cong E(\ln \widehat{P}_t - \ln \widehat{P}_{t-1}) \\ &= \sum_{i=0}^{n} \theta_i R_{t-i} \end{aligned}$$

He shows subsequently that adding leading and lagged market returns into the regression model can effectively solve the thin trading problem. However, two caveats we have to bear in mind when we apply this method to the stale pricing problem of private equity funds. First, Dimson replaces $\triangle \widehat{P}_t$ with $\widehat{R}_t$ in his derivation[10] to get the consistent aggregate beta. This is only legitimate when $\widehat{P}_t$ is very close to $\widehat{P}_{t-1}$ and P_t is very close to P_{t-1} in general. Severe biases occurs if the price has a trend or high variance due to the approximation of log-transformation. Since Dimson deals with daily data, it is reasonable to assume that the daily stock return has a mean zero and a small variance. But this is not the case for NAVs of private equity, which are reported on quarterly basis. It is unrealistic to assume that the NAV has a return of zero mean and small variance. We find that the bias increases with the mean and the variance of the stock in an unreported simulation. Although the stale pricing problem is similar to the thin trading problem, Dimson's method cannot be directly applied to estimate the systematic risk of private equity funds.

An additional issue, the fundamental assumption of price revelation is ad hoc here since observed prices have an expected value of past true prices. This implies, for example, the observed price at day t could be the true price of $t-1$,

[10] Dimson(1979), page 201.

but the observed price at day $t+1$ could be the true price of $t-3$.The observed prices can be inconsistent along the timeline by design. This assumption leads to a more parsimonious model as the consistency holds when n lagged market returns are included in the regression. However, this price revelation process is not in line with the observations we have on the stale NAVs.

Appendix 2: Why positive stock growth lead to a biased Dimson's aggregate beta

Suppose that the price discovery process for stock i is as follows:

$$E_t(\widehat{P}_{i,t}) = \theta P_{i,t} + (1-\theta)\widehat{P}_{i,t-1}$$

We assume that realization of the observed price at t will be either the true price at t or the observed price at $t-1$.The expectation of observed price at t then is simply the weighted average depending on the trading frequency (in terms of private equity fund, the NAVs update frequency), θ.The return then is as following

$$\begin{aligned} E_t(\Delta\widehat{P}_{i,t}) &= \theta\Delta P_{i,t} + (1-\theta)\Delta\widehat{P}_{i,t-1} \\ \frac{E_t(\Delta\widehat{P}_{i,t})}{\widehat{P}_{i,t-1}}\widehat{P}_{i,t-1} &= \theta\frac{\Delta P_{i,t}}{P_{i,t-1}}P_{i,t-1} + (1-\theta)\frac{\Delta\widehat{P}_{i,t-1}}{\widehat{P}_{i,t-2}}\widehat{P}_{i,t-2} \end{aligned}$$

Since at time t the observed price at $t-1$ is an realized value, so we can put $\widehat{P}_{i,t-1}$ into the expectation, and

$$E_t(\frac{\Delta\widehat{P}_{i,t}}{\widehat{P}_{i,t-1}})\widehat{P}_{i,t-1} = \theta\frac{\Delta P_{i,t}}{P_{i,t-1}}P_{i,t-1} + (1-\theta)\frac{\Delta\widehat{P}_{i,t-1}}{\widehat{P}_{i,t-2}}\widehat{P}_{i,t-2}$$

The observed return follows

$$\begin{aligned} E_t(\widehat{R}_{i,t})\widehat{P}_{i,t-1} &= \theta R_{i,t}P_{i,t-1} + (1-\theta)\widehat{R}_{i,t-1}\widehat{P}_{i,t-2} \\ E_t(\widehat{R}_{i,t}) &= \theta R_{i,t}\frac{P_{i,t-1}}{\widehat{P}_{i,t-1}} + (1-\theta)\widehat{R}_{i,t-1}\frac{\widehat{P}_{i,t-2}}{\widehat{P}_{i,t-1}} \\ \widehat{R}_{i,t} &= \theta R_{i,t}\frac{P_{i,t-1}}{\widehat{P}_{i,t-1}} + (1-\theta)\widehat{R}_{i,t-1}\frac{\widehat{P}_{i,t-2}}{\widehat{P}_{i,t-1}} + \nu_t \end{aligned}$$

where ν_t is mean zero error term. And replace $\widehat{R}_{i,t-1}$ with $\theta R_{i,t-1}\frac{P_{i,t-2}}{\widehat{P}_{i,t-2}} + (1-\theta)\widehat{R}_{i,t-2}\frac{\widehat{P}_{i,t-3}}{\widehat{P}_{i,t-2}} + \nu_{t-1}$, we have

$$
\begin{aligned}
\widehat{R}_{i,t} &= \theta R_{i,t}\frac{P_{i,t-1}}{\widehat{P}_{i,t-1}} + (1-\theta)[\theta R_{i,t-1}\frac{P_{i,t-2}}{\widehat{P}_{i,t-2}} + \\
&\quad (1-\theta)\widehat{R}_{i,t-2}\frac{\widehat{P}_{i,t-3}}{\widehat{P}_{i,t-2}} + \nu_{t-1}]\frac{\widehat{P}_{i,t-2}}{\widehat{P}_{i,t-1}} + \nu_t \\
&= \theta R_{i,t}\frac{P_{i,t-1}}{\widehat{P}_{i,t-1}} + (1-\theta)\theta R_{i,t-1}\frac{P_{i,t-2}}{\widehat{P}_{i,t-2}}\frac{\widehat{P}_{i,t-2}}{\widehat{P}_{i,t-1}} + \\
&\quad (1-\theta)^2\widehat{R}_{i,t-2}\frac{\widehat{P}_{i,t-3}}{\widehat{P}_{i,t-2}}\frac{\widehat{P}_{i,t-2}}{\widehat{P}_{i,t-1}} + (1-\theta)\nu_{t-1}\frac{\widehat{P}_{i,t-2}}{\widehat{P}_{i,t-1}} + \nu_t \\
&= \theta R_{i,t}\frac{P_{i,t-1}}{\widehat{P}_{i,t-1}} + (1-\theta)\theta R_{i,t-1}\frac{P_{i,t-2}}{\widehat{P}_{i,t-1}} + \\
&\quad (1-\theta)^2\widehat{R}_{i,t-2}\frac{\widehat{P}_{i,t-3}}{\widehat{P}_{i,t-1}} + (1-\theta)\nu_{t-1}\frac{\widehat{P}_{i,t-2}}{\widehat{P}_{i,t-1}} + \nu_t
\end{aligned}
$$

If we keep replacing $\widehat{R}_{i,t-k}$ with $\theta R_{i,t-k}\frac{P_{i,t-1-k}}{\widehat{P}_{i,t-1-k}} + (1-\theta)\widehat{R}_{i,t-1-k}\frac{\widehat{P}_{i,t-2-k}}{\widehat{P}_{i,t-1-k}} + \nu_{t-k}$ recursively and assume $\widehat{R}_0 = R_0$ and $\nu_0 = 0$, then

$$
\widehat{R}_{i,t} = \sum_{j=0}^{t}\theta(1-\theta)^j R_{i,t-j}\frac{P_{i,t-1-j}}{\widehat{P}_{i,t-1}} + \nu_t + \sum_{k=1}^{t}(1-\theta)^j\nu_{t-k}\frac{\widehat{P}_{i,t-1-k}}{\widehat{P}_{i,t-1}}
$$

Assuming a market model, $R_{i,t-j} = \alpha + \beta M_{t-j} + \varepsilon_{t-j}$ and substituting it into the equation, we have

$$
\begin{aligned}
\widehat{R}_{i,t} &= \sum_{j=0}^{t}\theta(1-\theta)^j[\alpha + \beta M_{t-j} + \varepsilon_{t-j}]\frac{P_{i,t-1-j}}{\widehat{P}_{i,t-1}} + \nu_t + \sum_{k=1}^{t}(1-\theta)^j\nu_{t-k}\frac{\widehat{P}_{i,t-1-k}}{\widehat{P}_{i,t-1}} \\
&= \sum_{j=0}^{t}\theta(1-\theta)^j\frac{P_{i,t-1-j}}{\widehat{P}_{i,t-1}}\alpha + \sum_{j=0}^{t}\theta(1-\theta)^j\frac{P_{i,t-1-j}}{\widehat{P}_{i,t-1}}\beta M_{t-j} + \\
&\quad \sum_{j=0}^{t}\theta(1-\theta)^j\frac{P_{i,t-1-j}}{\widehat{P}_{i,t-1}}\varepsilon_{t-j} + \nu_t + \sum_{k=1}^{t}(1-\theta)^j\nu_{t-k}\frac{\widehat{P}_{i,t-1-k}}{\widehat{P}_{i,t-1}}
\end{aligned}
$$

On the other hand, Dimson's aggregate beta model for thin trading problem is

$$
\widehat{R}_i = \alpha + \sum_{j=-n}^{n}\widehat{\beta}_j M_{t+j} + \upsilon_i
$$

Since the LHS of the aggregate model is identical with the model we derive from price discovery process, we can equate two M_{t-i} parts of the RHS, with the range from M_0 to M_t, to derive the relation of aggregate beta and the true beta:

$$\sum_{j=0}^{t} \theta(1-\theta)^j \frac{P_{i,t-1-j}}{\widehat{P}_{i,t-1}} \beta = \sum_{j=0}^{t} \widehat{\beta}_{-j}$$

or to be more illustrative

$$[\theta \frac{P_{i,t-1}}{\widehat{P}_{i,t-1}} + (1-\theta)\theta \frac{P_{i,t-2}}{\widehat{P}_{i,t-1}} + (1-\theta)^2 \theta \frac{P_{i,t-3}}{\widehat{P}_{i,t-1}} ...]\beta = \sum_{j=0}^{t} \widehat{\beta}_{-j}$$

Note: We have shown that aggregate beta is biased due to the non-zeros drift of RV and one cannot include all the lagged terms. The estimate so far is based on conditional return. When calculate the alpha, we need the unconditional return since

$$\widehat{\alpha} = E(\widehat{R}_{i,t}) - \widehat{\beta} E(M_{,t})$$

But Brennan and Wang (2006) show that the unconditional return, $E(\widehat{R}_{i,t})$, is also biased. This is because price is a non-linear function of expected return, so that if one variable is subject to random error then the expectation of the other variable will be biased.

Table 1: Descriptive Statistics

This table gives descriptive statistics of two samples as of December 2003. Statistics for venture and buyout funds within each sample are reported separately. We report, respectively and for each sample: (i) the average (equal weights) and median of the amount invested by funds in millions of dollars (Invested); (ii) the proportion of first time funds; (iii) the proportion of non-US investments (in number) ; (iv) average numbers of dividend payout per fund; (v) frequency of dividend payout per quarter. The last two rows are the numbers of cash flows and the numbers of funds. The two samples consist of: First, the universe of funds in Venture Economics raised between 1980 and 2003. Second, the quasi-liquidated funds raised between 1980 and 1996. A fund is considered quasi-liquidated if it has cash-flow information and is either officially liquidated or has no cash-flow from July 2002 to December 2003. We also eliminate funds that pay no dividend at all in the quasi-liquidated sample.

	Full dataset			Quasi-liquidated		
	All	VC	BO	All	VC	BO
Mean Invested (mn. Dec 2003)	325	163	623	135	79	276
Median Invested (mn. Dec 2003)	101	70	259	59	48	126
First time (%)	0.41	39	33	37	35	41
Non-US (%)	0.35	32	40	29	22	46
Mean numbers of divd payout	10	9	11	15	14	16
Mean freq. of divd. payout (%)	34	29	42	40	37	47
Numbers of cash flows	52,891	30,758	22,133	23,296	15,731	7,565
Numbers of funds	2,420	1,570	850	941	673	268

Table 2: Results for simulated private equity economy, Small Sample

This table reports the summary of calibration. We choose the calibration such that the result is closest to the real data at hand in terms of the fraction of funds liquidated, average age of funds at liquidation and average numbers of dividend payouts. The magnitude of the idiosyncratic shock is calculated by the back-of-the-envelope, assuming PE funds have 40% annual volatility.

Panel A: Calibrations

Number of periods:	100 quarters (i.e. 25 years)
Number of funds:	20 funds per vintage year
	(for the first 15 years; total 300 funds)
Number of simulations:	1000 simulations
Alpha:	1% (per quarter)
Beta:	1.5
Risk-free rate:	1% (per quarter)
Market expected return:	3% (per quarter)
Market volatility (standard deviation):	10% (per quarter)
Starting value of fund:	1
Liquidation threshold:	0.1
Dividend payout ratio:	0.2
Frequency of dividend payout:	0.25
Frequency of revealed value:	0.125
Idiosyncratic shock:	0.13

Panel B: Stale Pricing Without Idiosyncratic Shocks

Frequency of revealed value: average every 2 years

	Agg.β with 4 lags		Agg.β with 8 lags		GMM	
	Alpha	Beta	Alpha	Beta	Alpha	Beta
Mean	1.19%	0.75	0.47%	1.17	1.04%	1.49
Median	1.26%	0.74	0.53%	1.16	1.02%	1.50
Min	-2.22%	0.51	-2.33%	0.70	0.82%	1.03
Max	3.94%	1.28	2.14%	1.95	2.17%	1.61
Std	0.91%	0.10	0.62%	0.13	0.08%	0.03

Extra statistics:

Fraction liquidated	0.31
Mean age at liquidation	51.13
Nbs of dividends	15.92

Panel C: Stale Pricing With Idiosyncratic Shocks

Frequency of revealed value: average every 2 years						
	Agg.β with 4 lags		Agg.β with 8 lags		GMM	
	Alpha	Beta	Alpha	Beta	Alpha	Beta
Mean	1.19%	0.77	0.47%	1.19	0.92%	1.51
Median	1.24%	0.76	0.48%	1.18	0.91%	1.52
Min	-1.62%	0.51	-1.53%	0.82	-0.14%	0.93
Max	3.86%	1.20	2.38%	1.82	2.34%	2.41
Std	0.89%	0.11	0.61%	0.14	0.26%	0.16

Extra statistics:	
Fraction liquidated	0.48
Mean age at liquidation	44.80
Nbs of dividends	14.20

Table 3: Results for simulated private equity economy, Large Sample

The table reports the result when we enlarge the sample size in terms of periods, vintage years and the fund numbers. By increasing the sample size, we can see that the GMM model has a nice asymptotic property while the JRK's method remains biased and inconsistent.

Panel A: Calibrations

Number of periods:	500 quarters (i.e. 125 years)
Number of funds:	15 funds invested per vintage year
	(for the first 100 years; total 1500 funds)

Panel B: Stale Pricing Without Idiosyncratic Shocks

Frequency of revealed value: average every 2 years						
	Agg.β with 4 lags		Agg.β with 8 lags		GMM	
	Alpha	Beta	Alpha	Beta	Alpha	Beta
Mean	0.98%	0.80	0.20%	1.25	1.00%	1.50
Median	0.99%	0.80	0.22%	1.25	1.00%	1.50
Min	-0.16%	0.67	-0.55%	1.07	1.00%	1.50
Max	2.05%	1.00	0.85%	1.47	1.00%	1.50
Std	0.36%	0.05	0.23%	0.06	0.00%	0.00

Panel C: Stale Pricing With Idiosyncratic shocks

Frequency of revealed value: average every 2 years						
	Agg.β with 4 lags		Agg.β with 8 lags		GMM	
	Alpha	Beta	Alpha	Beta	Alpha	Beta
Mean	1.02%	0.80	0.24%	1.25	0.88%	1.51
Median	1.01%	0.80	0.24%	1.25	0.88%	1.51
Min	-0.03%	0.56	-0.72%	1.02	0.67%	1.38
Max	2.85%	1.10	1.99%	1.60	1.12%	1.68
Std	0.37%	0.06	0.24%	0.07	0.07%	0.04

Table 4: Biased Aggregate Beta When Stock has Positive Drift

This table shows that Dimson's aggregate beta method can not be applied if stock price has a positive drift. The simulation setup is as following: the true price follows CAPM with alpha equals to zero and beta equals to one. The market excess return follows a normal distribution with mean μ and variance 0.01.

$$P_{i,t}= P_{i,t-1}(1 + r_{m,t}) \quad \text{where} \quad r_m \sim N(\mu, 0.01)$$

The price revelation price is

$$E(\widehat{P}_{i,t}) = \theta P_{i,t} + (1 - \theta)\widehat{P}_{i,t-1}$$

that realization of the observed price at t will be either the true price at t or the observed price at $t - 1$.The expectation of observed price at t then is simply the weighted average depending on the revelation frequency θ. Column 3, 4, 5, 6 and 7 report the aggregate betas with 0, 4, 8, 20 and 50 lags respectively. We run 1000 simulations with t equals to 10000 and with three different μ. We report both the mean and the median of the aggregate beta.

Frequency of revealed value: average every 2 years						
		β_0	$\sum_{i=0}^{4}\beta_i$	$\sum_{i=0}^{8}\beta_i$	$\sum_{i=0}^{20}\beta_i$	$\sum_{i=0}^{50}\beta_i$
$\mu = 0$	mean	0.13	0.49	0.70	0.94	1.00
	median	0.12	0.49	0.70	0.94	0.99
$\mu = 1\%$	mean	0.13	0.53	0.77	1.07	1.15
	median	0.14	0.53	0.77	1.07	1.14
$\mu = 2\%$	mean	0.14	0.58	0.86	1.23	1.35
	median	0.15	0.58	0.86	1.23	1.34

Table 5

The sample is all quasi-liquidated funds, which are funds raised between 1980 and 1996 with no cash flows activities between July 2002 to December 2003. The first column is result of CAPM. The second and third column add Fama-French's SMB, HML and Pástor and Stambaugh's liquidity factors. The fourth to sixth columns consists the results of time-varying beta, where β_1 is the time-varying components. We bootstrap 1,000 times for getting the standard errors. ***, **, * indicate significant level at 1%, 5% and 10% .

Panel A: Result for quasi-liquidated funds, Equally Weighted

	CAPM	FF	FF+LIQ	Call	Call+FF	Call +FF+LIQ
α (%)	-0.73***	-0.46***	-0.49***	-0.54***	-0.17	-0.16
	(0.11)	(0.18)	(0.18)	(0.11)	(0.21)	(0.22)
β_0	1.05***	1.08***	1.09***	1.64***	1.47***	1.49***
	(0.20)	(0.21)	(0.22)	(0.22)	(0.24)	(0.25)
β_1				-0.72***	-0.72***	-0.74***
				(0.10)	(0.11)	(0.11)
SMB		0.61***	0.72***		0.29	0.45*
		(0.21)	(0.24)		(0.22)	(0.27)
HML		-0.58**	-0.61**		-0.73***	-0.85***
		(0.29)	(0.28)		(0.26)	(0.25)
LIQ			0.16			0.24*
			(0.12)			(0.13)
Mean sqr. error	0.623	0.614	0.613	0.607	0.601	0.599
Nb. of Funds						941

Panel B: Result for quasi-liquidated VC funds, Equally Weighted

	CAPM	FF	FF+LIQ	Call	Call+FF	Call +FF+LIQ
α (%)	-0.86***	-0.46**	-0.47**	-0.65***	-0.14	-0.14
	(0.14)	(0.21)	(0.21)	(0.15)	(0.24)	(0.25)
β_0	1.23***	1.28***	1.28***	1.89***	1.68***	1.69***
	(0.28)	(0.27)	(0.28)	(0.31)	(0.31)	(0.33)
β_1				-0.79***	-0.75***	-0.77***
				(0.13)	(0.15)	(0.14)
SMB		0.74***	0.78***		0.62**	0.70**
		(0.26)	(0.28)		(0.27)	(0.29)
HML		-0.62*	-0.63*		-0.95***	-0.99***
		(0.35)	(0.33)		(0.26)	(0.26)
LIQ			0.05			0.12
			(0.11)			(0.14)
Mean sqr. error	0.681	0.662	0.662	0.662	0.648	0.647
Nb. of Funds						673

Panel C: Result for quasi-liquidated BO funds, Equally Weighted

	CAPM	FF	FF+LIQ	Call	Call+FF	Call +FF+LIQ
α (%)	-0.38**	-0.47*	-0.69**	-0.31	-0.27	-0.48
	(0.17)	(0.28)	(0.35)	(0.19)	(0.36)	(0.46)
β_0	0.66**	0.58**	0.67*	1.04***	0.96***	0.99**
	(0.26)	(0.29)	(0.34)	(0.26)	(0.33)	(0.39)
β_1				-0.45*	-0.55***	-0.52**
				(0.24)	(0.19)	(0.21)
SMB		-0.86	-1.02		-0.66**	-0.88
		(1.05)	(1.61)		(0.27)	(0.63)
HML		1.18	1.29		0.64	0.84
		(0.91)	(1.18)		(0.44)	(0.55)
LIQ			1.19			0.77**
			(1.05)			(0.34)
Mean sqr. error	0.463	0.456	0.438	0.459	0.448	0.432
Nb. of Funds						268

Table 6

The sample is all quasi-liquidated funds, which are funds raised between 1980 and 1996 with no cash flows activities between July 2002 to December 2003. The first column is result of CAPM. The second and third column add Fama-French's SMB, HML and Pástor and Stambaugh's liquidity factors. The fourth to sixth columns consists the results of time-varying beta, where β_1 is the time-varying components. We bootstrap 1,000 times for getting the standard errors. ***, **, * indicate significant level at 1%, 5% and 10% .

Panel A: Result quasi-liquidated funds, Value Weighted

	CAPM	FF	FF+LIQ	Call	Call+FF	Call +FF+LIQ
α (%)	-0.79***	-0.53***	-0.56***	-0.62***	-0.24	-0.25
	(0.12)	(0.18)	(0.18)	(0.12)	(0.22)	(0.23)
β_0	1.14***	1.14***	1.16***	1.72***	1.53***	1.57***
	(0.21)	(0.22)	(0.22)	(0.23)	(0.25)	(0.26)
β_1				-0.70***	-0.70***	-0.73***
				(0.11)	(0.13)	(0.13)
SMB		0.49**	0.58**		0.21	0.34
		(0.21)	(0.24)		(0.23)	(0.28)
HML		-0.53*	-0.55*		-0.71***	-0.80***
		(0.29)	(0.28)		(0.27)	(0.27)
LIQ			0.15			0.22
			(0.13)			(0.15)
Nb. of Funds						941

Panel B: Result of quasi-liquidated VC funds, Value Weighted

	CAPM	FF	FF+LIQ	Call	Call+FF	Call +FF+LIQ
α (%)	-0.81***	-0.44**	-0.45**	-0.60***	-0.09	-0.10
	(0.14)	(0.21)	(0.21)	(0.15)	(0.25)	(0.26)
β_0	1.12***	1.18***	1.18***	1.82***	1.63***	1.63***
	(0.28)	(0.27)	(0.27)	(0.31)	(0.31)	(0.33)
β_1				-0.81***	-0.80***	-0.81***
				(0.13)	(0.21)	(0.19)
SMB		0.79***	0.81***		0.60**	0.66**
		(0.27)	(0.28)		(0.29)	(0.30)
HML		-0.67**	-0.67***		-0.95***	-0.98***
		(0.34)	(0.33)		(0.27)	(0.27)
LIQ			0.04			0.10
			(0.12)			(0.15)
Nb. of Funds						673

Panel C: Result of quasi-liquidated BO funds, Value Weighted

	CAPM	FF	FF+LIQ	Call	Call+FF	Call +FF+LIQ
α (%)	-0.33*	-0.40	-0.60*	-0.27	-0.25	-0.44
	(0.18)	(0.28)	(0.34)	(0.20)	(0.36)	(0.46)
β_0	0.57**	0.48*	0.57*	0.91***	0.83**	0.92**
	(0.27)	(0.29)	(0.33)	(0.26)	(0.33)	(0.40)
β_1				-0.39	-0.47**	-0.49**
			()	(0.25)	(0.20)	(0.23)
SMB		-1.04	-1.22		-0.63**	-0.87
		(1.00)	(1.58)		(0.27)	(0.65)
HML		1.34	1.38		0.59	0.75
		(1.08)	(1.30)		(0.44)	(0.55)
LIQ			1.45			0.83**
			(1.05)			(0.36)
Nb. of Funds						268

Table 7

The sample is all quasi-liquidated funds, which are funds raised between 1980 and 1993 with no cash flows activities between July 2002 to December 2003. The first column is result of CAPM. The second and third column add Fama-French's SMB, HML and Pástor and Stambaugh's liquidity factors. The fourth to sixth columns consists the results of time-varying beta, where β_1 is the time-varying components. We bootstrap 1,000 times for getting the standard errors. ***, **, * indicate significant level at 1%, 5% and 10% .

Panel A: Results for subsample of quasi-liquidated funds, Equally Weighted

	CAPM	FF	FF+LIQ	Call	Call+FF	Call +FF+LIQ
α (%)	-0.83***	-0.79***	-0.80***	-0.65***	-0.58**	-0.59**
	(0.17)	(0.25)	(0.28)	(0.18)	(0.26)	(0.30)
β_0	1.15***	1.18***	1.17***	1.50***	1.46***	1.45***
	(0.30)	(0.30)	(0.28)	(0.29)	(0.32)	(0.31)
β_1				-0.53***	-0.54**	-0.53***
				(0.13)	(0.22)	(0.19)
SMB		0.19	0.12		-0.05	-0.11
		(0.30)	(0.65)		(0.29)	(0.63)
HML		-0.13	-0.05		-0.12	0.00
		(0.39)	(0.61)		(0.37)	(0.61)
LIQ			-0.04			-0.09
			(0.30)			(0.30)
Mean sqr. error	0.578	0.578	0.578	0.571	0.571	0.571
Nb. of Funds						816

Panel B: Results for subsample of quasi-liquidated VC funds, Equally Weighted

	CAPM	FF	FF+LIQ	Call	Call+FF	Call +FF+LIQ
α (%)	-1.01***	-0.71**	-0.45	-0.83***	-0.59	-0.68*
	(0.21)	(0.39)	(0.35)	(0.25)	(0.40)	(0.39)
β_0	1.43***	1.45***	1.47***	1.76***	1.69***	1.69***
	(0.42)	(0.37)	(0.34)	(0.41)	(0.45)	(0.39)
β_1				-0.52**	-0.49	-0.50
				(0.25)	(0.56)	(0.38)
SMB		-0.46	0.98		0.35	0.04
		(0.62)	(0.70)		(0.63)	(0.86)
HML		0.62	-0.72		-0.55	-0.14
		(0.57)	(0.65)		(0.50)	(0.79)
LIQ			0.18			-0.20
			(0.33)			(0.38)
Mean sqr. error	0.601	0.596	0.594	0.595	0.593	0.592
Nb. of Funds						590

Panel C: Results for subsample of quasi-liquidated BO funds, Equally Weighted

	CAPM	FF	FF+LIQ	Call	Call+FF	Call +FF+LIQ
α (%)	-0.33	-0.56*	-0.67*	-0.25	-0.42	-0.53
	(0.26)	(0.30)	(0.38)	(0.29)	(0.41)	(0.51)
β_0	0.57	0.55*	0.54	0.86**	0.85**	0.87*
	(0.37)	(0.33)	(0.38)	(0.34)	(0.41)	(0.48)
β_1				-0.38	-0.41*	-0.43*
				(0.28)	(0.21)	(0.22)
SMB		-1.18	-1.61		-0.72***	-0.93
		(1.17)	(2.32)		(0.27)	(0.89)
HML		2.25	2.60		1.14**	1.29*
		(1.97)	(2.23)		(0.51)	(0.68)
LIQ			1.21			0.67
			(1.23)			(0.44)
Mean sqr. error	0.499	0.481	0.468	0.496	0.477	0.464
Nb. of Funds						226