Correlation Risk and Optimal Portfolio Choice

Andrea Buraschi

(with Fabio Trojani and Paolo Porchia)

Stylized Example: A Convergence Trade

- A market neutral hedge fund arbitraging yield differences between 10 year Treasury bills and corporate Aaa bonds:.

$$Y_{Tr} = 5.45 \qquad Y_{Aaa} = 6.01$$

- Suppose in the last 5 years the correlation has been 0.97. You might be tempted to implement a convergence trade to take advantage of the positive carry => LTCM.

- Questions:

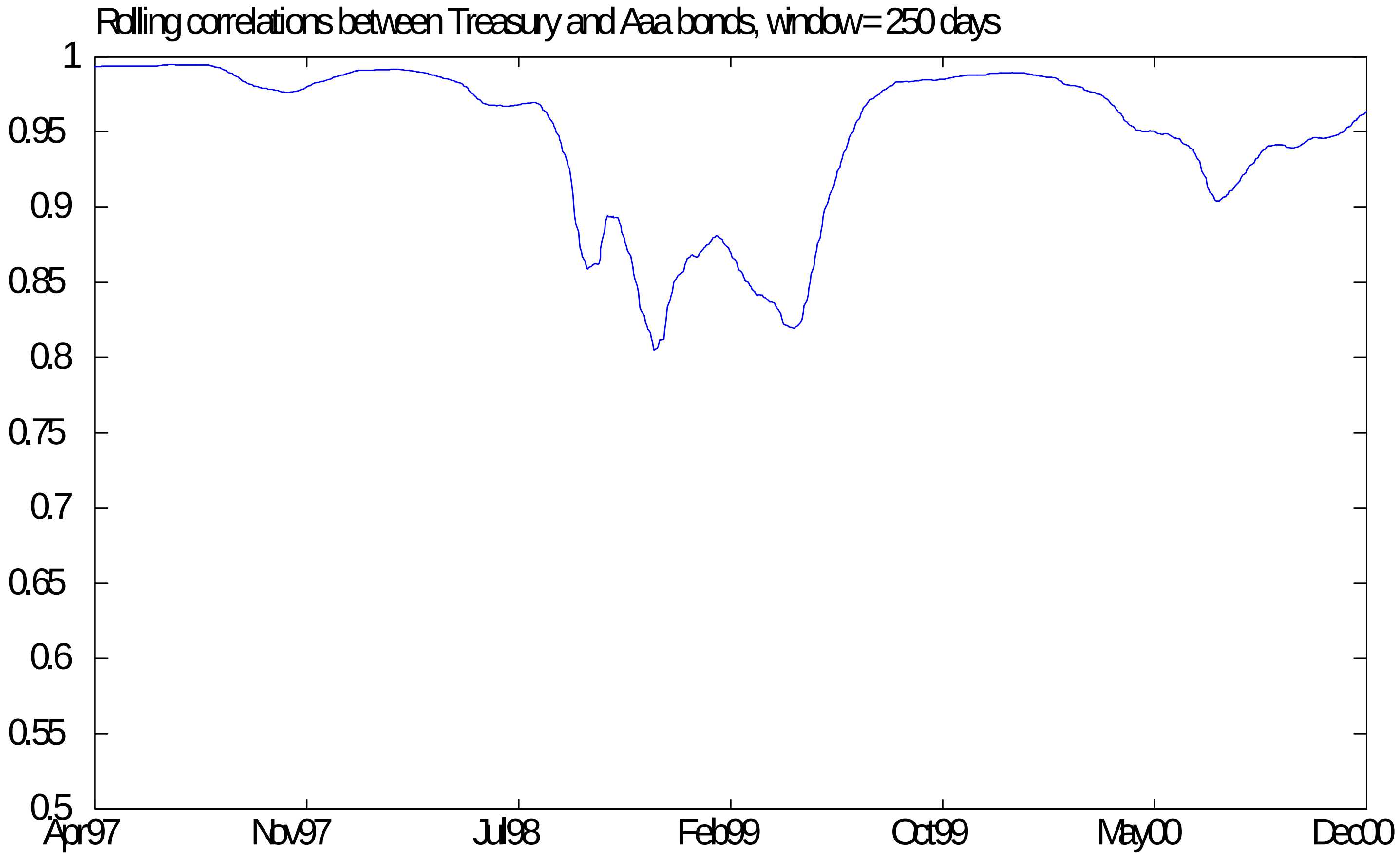
1. What is the average correlation between Treasury and Aaa corporate bond yields?
2. Is the correlation between Treasury and Aaa corporate bond yields constant?

Stylized Example: A Convergence Trade

- Average Correlation: $\rho = 0.97$ before 1998.
- Time Variation $std(\rho) = 0.10!$
- After Russian default ρ dropped to 0.80!
- *Implications for dynamic portfolio choice?*

==> Clearly massive (ex-post)

=> What about ex-ante (hedging component)?


Rolling correlations between Treasury and Aaa bonds, window=250 days
1
0.95
0.9
0.85
0.8
0.75
0.7
0.65
0.6
0.55
0.5
Apr97
Nov97
Jul98
Feb99
Oct99
May00
Dec00

Pension Funds and Hedge Funds

- Question: Who is the biggest holder of hedge funds? Who buys them?

Pension Funds and Hedge Funds

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- Answer: *Pension Funds!*
- In 2005, pension funds globally invested more than \$77Bl of new money in alternative assets, up from \$62Bl in 2004. New cash inflows: 34% invested in funds of hedge funds.
- Retirement funds are the single largest holder of alternative assets, accounting for about \$465Bl, or 37%.
  - Late 1990s, these strategies paid off and fund surpluses grew to about 115%.
  - After the bubble burst, U.S. funding ratios declined to about 80%.
  - This creates a very bad incentive: risk taking strategies!
- Bottom line: correlation risk is much more general than previously thought*!*

Pension Funds

- Define Benefits pension funds have inflation-linked liabilities.

- Suppose you are a pension fund manager. What would you do if:
  1. You want to minimize the risk at maturity
  2. You are not interested in adding a return.

Pension Funds

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- Suppose you are a pension fund manager. What would you do if:

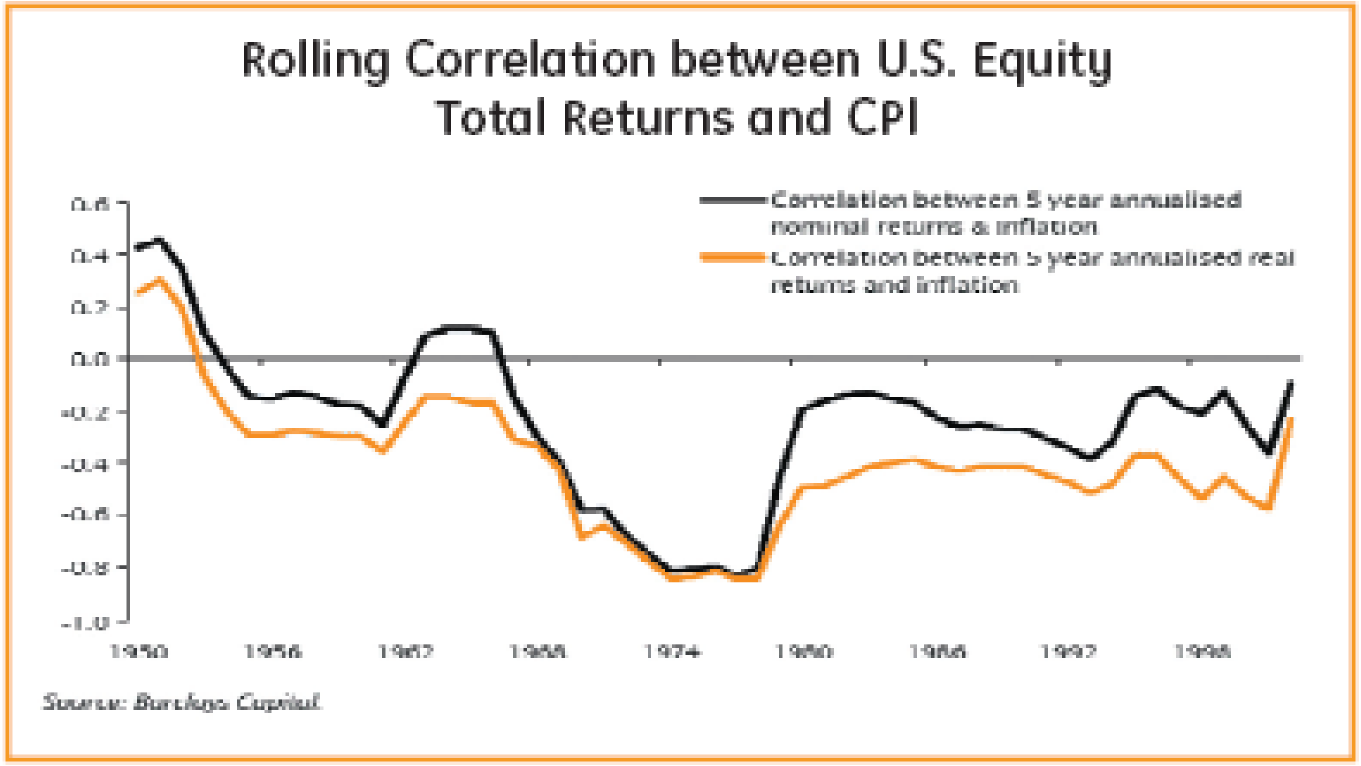
  1. You want to minimize the risk at maturity
  2. You are not interested in adding a return.

- Buy Equity! They are a good protection against inflation.

Pension Funds

- Does it really work?

- Sorry.. Not quite .....



- Substantial correlation risk.

Index-Linked Bonds

- Pension Funds have a much better idea! What about using Index-Linked Bonds (in duration matching strategy)?

- Would this strategy work?

- Yes, if Indexed-Linked bonds have $\rho = 1$ with inflation

Rolling 12-month Correlations (monthly data)
Feb 1997 - Feb 2005

	10yr UST	10yr TIPS	CPI	SP500
10yr UST	1.00			
10yr TIPS	0.58	1.00		
CPI	-0.20	0.34	1.00	
SP500	-0.38	-0.72	-0.22	1.00

Source: Barclays Capital, Bloomberg

- Well.... they work better than equities, but:
  - Correlation is far from $\rho = 1$, and time-varying!

- The results is true for almost any country and at several maturities:

TABLE ONE: DOMESTIC INFLATION-LINKED BONDS VS INTERNATIONAL INFLATION-LINKED BONDS

	Correlation to Domestic CPI Inflation over:				
Correlation of:	**1mth**	**3mth**	**12mth**	**3yr**	**5yr**
Australia					
Domestic I/I Portfolio	0.05	0.25	0.40	0.58	0.69
Global I/I Portfolio	0.08	0.32	0.52	0.71	0.80
Canada					
Domestic I/I Portfolio	0.20	0.23	0.31	0.46	0.61
Global I/I Portfolio	0.13	0.23	0.44	0.64	0.75
Euroland (France)					
Domestic I/I Portfolio	0.22	0.31	0.50	0.76	0.87
Global I/I Portfolio	0.16	0.27	0.50	0.72	0.82
UK					
Domestic I/I Portfolio	0.17	0.32	0.68	0.82	0.91
Global I/I Portfolio	0.15	0.29	0.60	0.83	0.90
US					
Domestic I/I Portfolio	0.13	0.20	0.31	0.41	0.52
Global I/I Portfolio	0.11	0.20	0.37	0.59	0.71
Average					
Domestic I/I Portfolio	0.15	0.26	0.44	0.61	0.72
Global I/I Portfolio	0.13	0.26	0.49	0.70	0.80

Source: Bridgewater

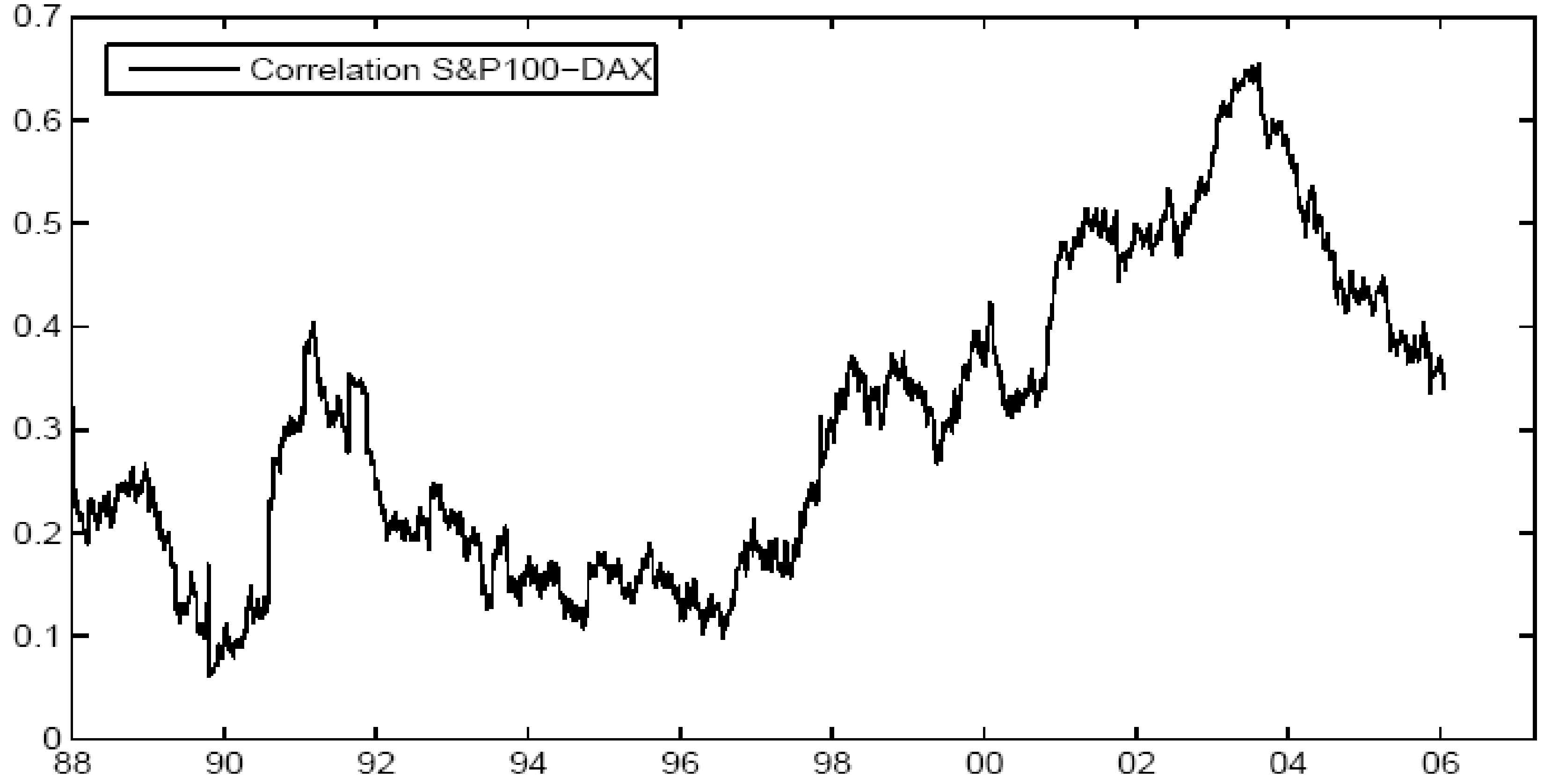
**Global I/I portfolios consist of Australia (10%), Canada (10%), Euroland (30%), UK (15%) and US (35%). 10-year duration inflation-linked bonds hedged to the respective local currency. Figures based on simulated and real performance history from January 1970 to April 2005*

- Bottom Line: Even if you stay away from hedge funds and use "safe" index-linked bonds, you will incur substantial correlation risk.

Really lots of other examples of "correlation risk" ...

- A internation equity mutual fund the (SP500-Dax $std(\rho) = 0.15$).
- A long-short strategy on the spread between On-the-run versus Off-the-run bonds.
- A long-short strategy on the spread between Shell and Royal Dutch.
- Capital structure arbitrage trades, etc...

Implications for dynamic portfolio choice?


Correlation S&P100−DAX
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
88
90
92
94
96
98
00
02
04
06

Empirical Evidence

- **Var and Covariances are time varying:** Large (Garch) literature on asset returns that exhibit time-varying variance and covariances (e.g. Bollerslev, Engle and Wooldridge, 1988, and Bollerslev, Chou and Kroner, 1992).

Time variation derives from changing volatilities and *correlations*.

- **Correlation risk is economically relevant:** Moskovitz (2003) [*pricing anomalies*]; Driessen and Maenhout (2003) [*option pricing*].

- **Correlation risk and Business Cycle:** Ang and Chen (2001): correlation between US stocks and the aggregate market is much higher during extreme downside movements

- **International Markets:** Erb, Harvey and Viskanta (1994): countries are more correlated during recessionary states.

Question: *does this empirical evidence have quantitative implications for dynamic portfolio choice?*

Goal of the Paper

- Take seriously into account the stochastic nature of correlation and study its implications in optimal portfolio choice.
  - Multivariate framework
  - Incomplete markets economy due to stoch corr
  - Restrictions: Corr must satisfy some specific properties in order to be consistent with a positive definite cov matrix

- Results:
  1. An additional hedging demand for correlation risk emerges.
  2. Small changes in correlations might imply large changes in investors' marginal utilities.
  3. Obtain closed-form solution for optimal portfolios
  4. We can investigate qualitatively and quantitatively the size of the intertemporal correlation hedging demand!

Research Agenda

- Part of a more general research agenda:

1. Equilibrium asset pricing with correlation risk: "the price of correlation risk".
2. Modeling correlation derivatives (quanto futures, differential swaps).
3. Modeling correlation risk in CDOs (avoiding the assumptions implicit in Gaussian copulas)
4. Term structure modeling with stochastically correlated factors.

Related Literature

- **Univariate portfolio choice:** Kim and Omberg (1996), Wachter (2001) [*stochastic Sharpe-ratio*]; Chacko and Viceira (2005) [*stochastic volatility*].

- **Gaussian multivariate portfolio choice:** Brennan and Xia (2002) [*inflation risk*]; Sangvinatos and Wachter (2005) [*long run bond investment*]; Mulk and Sorensen (2004) [*bond investment with non Markovian interest rates*] .

  Returns covariance matrix is deterministic!

- **Multivariate portfolio choice with stochastic volatility:** Liu (2006) [*cash-stock-bond allocation*] .

  No separate role for stochastic correlations and correlation hedging!

The Model

- **Investment universe:** cash with riskless return $r \in \mathbb{R}^+$ and two risky assets with price process $S = (S_1, S_2)'$, having dynamics

$$dS(t) = I_S \left[(r + \Sigma(t)\lambda)dt + \Sigma(t)^{1/2} dW(t) \right] \quad ; \quad I_S = diag[S_1, S_2]$$

where $\lambda \in \mathbb{R}^2$, $\Sigma(t)$ is a stochastic covariance matrix and $W = (W_1, W_2)'$ is a standard BM.

- **Wealth dynamics:** For fractions $\pi = (\pi_1, \pi_2)'$ allocated to risky assets, wealth $X(t)$ has dynamics

$$dX(t) = X(t) \left[r + \pi(t)'\Sigma(t)\lambda \right] dt + X(t)\pi(t)'\Sigma^{1/2}(t)dW(t)$$

The Covariance Process

- **Wishart process:** Given square matrices Ω, M, Q, we posit a variance covariance matrix process with dynamics:

$$d\Sigma(t) = \left[\Omega\Omega' + M\Sigma(t) + \Sigma(t)M'\right] dt + \Sigma^{1/2}(t)\underbrace{dB(t)}_{2x2}Q + Q'dB(t)'\Sigma^{1/2}(t)$$

  where $B = [W, Z]$ is a square matrix of orthogonal standard Brownian motions.

- Statistical properties of this (affine) process has been studied by Gourieroux in a series of papers: Gourieroux and Sufana (2004), Gourieroux, Jasiak, and Sufana (2004).

- **Features:**
  1. Σ follows a genuine stochastic volatility and correlation process (unlike Garch) and implies a Wishart distribution for $\Sigma(t)$. In univariate case => Heston (1993).
  2. Σ satisfies the conditions for being symmetric and positive-definite.
  3. With $B = [W, Z]$, we model easily (directly) the "leverage effect" (e.g. q_{11} drives the Asset 1 vol-leverage)
  4. Note that since $E(dBdZ) = 0$, the assets do not span the state-space: incomplete markets.

Incomplete Markets

- Consider a fixed investment horizon $T > 0$. If X_0 is the initial wealth, and Σ_0 denotes the initial covariance matrix, the investor's optimization problem is:

$$J(X_0, \Sigma_0) = \sup_{\pi} \mathbb{E}\left[\frac{X(T)^{\gamma} - 1}{\gamma}\right],$$

subject to the dynamic budget constraint.

- Since stochastic correlation implies incomplete markets => multiple kernels.

- Among the multiple pricing kernels, we choose min-max martingale measure (He and Pearson (1991)):

$$J(X_0, \Sigma_0) = \inf_{\nu} \sup_{\pi} \mathbb{E}\left[\frac{X(T)^{\gamma} - 1}{\gamma}\right]$$
$$\text{s.t.} \quad \mathbb{E}\left[\xi_{\nu}(T)X(T)\right] \leq x$$

where ν indexes the set of all equivalent martingale measures in our model and ξ_{ν} is in the set of associated state price densities.

Modeling Issues in Previous Models

- In the term structure literature, modelling dependence in factor models is not trivial if you want closed-form solutions.

- Duffie and Kan (1996):
  - Parametric restrictions on the drift matrix (factors) => Ensure existence of regular affine process (e.g. off-diagonal elements must have the same sign). Duffee (2002) discusses empirical implications of these restrictions.
  - Square-root factors cannot be correlated for closed-form solutions.

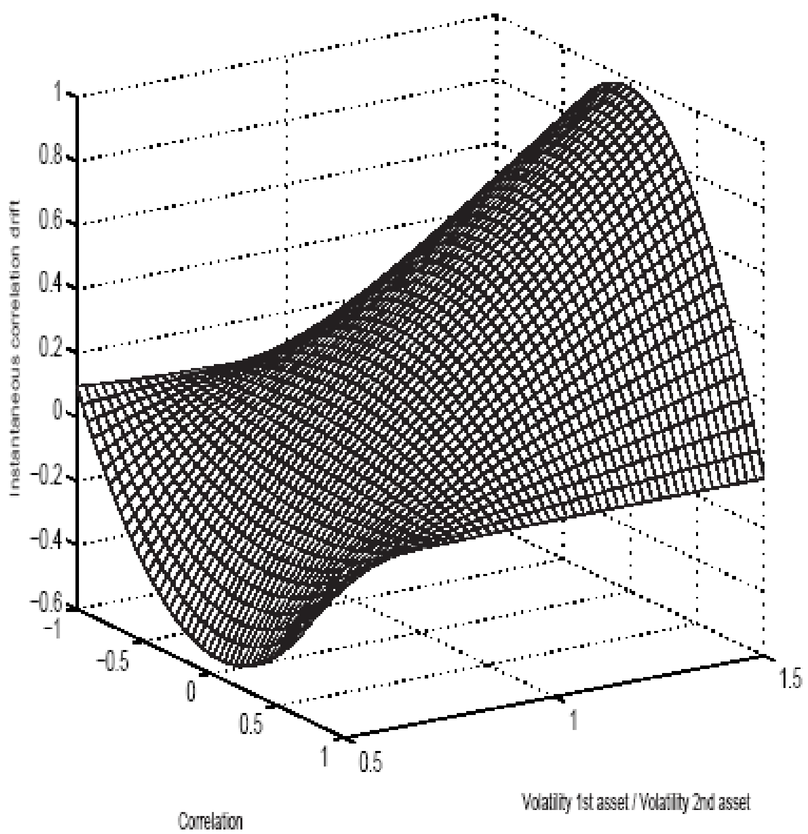
- Alternatives:
  - *Driessen and Maenhout* (2005) specify the process for the average correlation as $d\rho = k(\rho - \bar{\rho})dt + \sigma\sqrt{\rho(1-\rho)}dW$.Problematic to generalize it to multiple ρ and for an optimal portfolio problem).
  - *Harvey, Ruiz, Shephard* (1994):

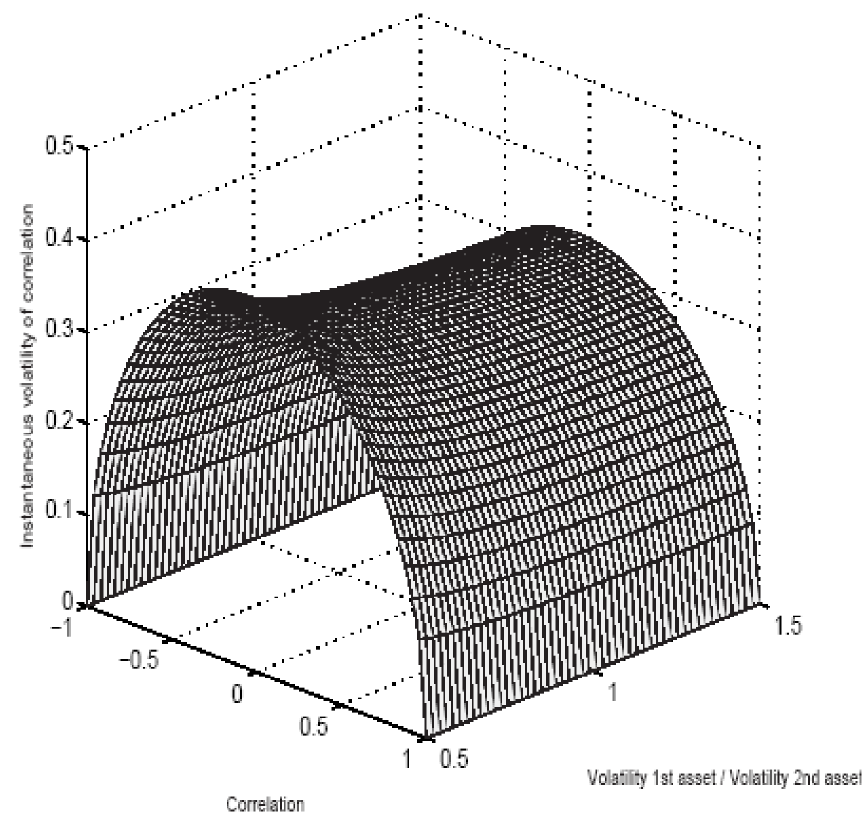
$$\Sigma = A \begin{pmatrix} \exp h_1(t) & & \\ & \exp h_3(t) & \\ & & \exp h_3(t) \end{pmatrix} A'$$

    $h_i(t)$ are the n factors for the $n \times n$ matrix Σ. Cannot model the correlation independently: we need $n + n(n-1)/2$ elements => Curse of dimensionality.
  - *Liu* (2005) uses a string aproach to model the elements of $\Sigma_{Xij}(X) = h_{0ij} + h'_{1ij}X + X'\eta' h_{2ij}\eta X$. The conditions for positive definiteness are state dependent. Difficult to isolate the component due to correlation.

Implied Correlation Drift and Local Vol

- Use Ito's Lemma and compute implied process for correlation.
- It is non-linear: Drift is quadratic, local volatility is cubic: even if covariance process is affine.
- Different, for instance, from Driessen and Maenhout (2005)





Solution of the Portfolio Problem

- Optimal portfolio:
  - Myopic Merton portfolio
  - Hedging demand for stochastic volatility
  - Hedging demand for correlation risk.

$$\pi = \underbrace{\frac{\lambda}{1-\gamma}}_{Myopic} + \underbrace{\begin{pmatrix} q_{11}\left(\overline{A}_{11} + \overline{A}_{12}\overline{\rho}\sqrt{\frac{\Sigma_{11}}{\Sigma_{22}}}\right) \\ q_{12}\left(\overline{A}_{22} + \overline{A}_{12}\overline{\rho}\sqrt{\frac{\Sigma_{22}}{\Sigma_{11}}}\right) \end{pmatrix}}_{Volatility\ Hedging} + \underbrace{\begin{pmatrix} q_{12}A_{12} - q_{11}\overline{A}_{12}\overline{\rho}\sqrt{\frac{\Sigma_{11}}{\Sigma_{22}}} \\ q_{11}A_{12} - q_{12}\overline{A}_{12}\overline{\rho}\sqrt{\frac{\Sigma_{22}}{\Sigma_{11}}} \end{pmatrix}}_{Correlation\ Hedging}$$

Calibration: Hedging Portfolios

Two Scenarios:

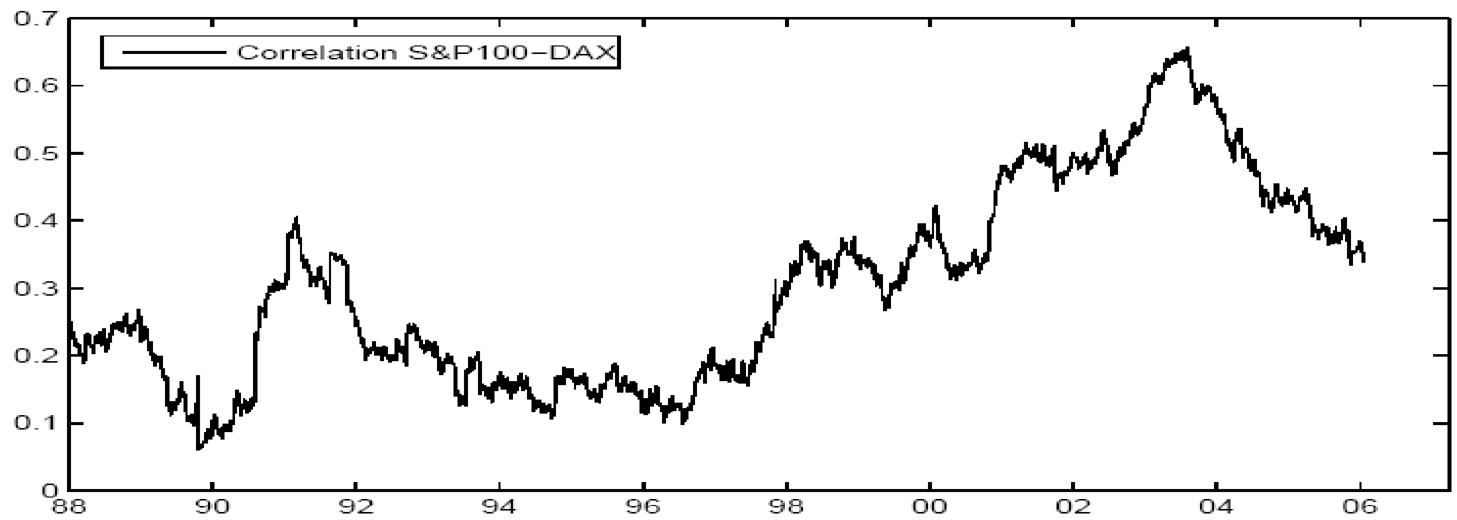
CASE 1. Simplified international portfolio allocation problem where average correlations are not extreme.

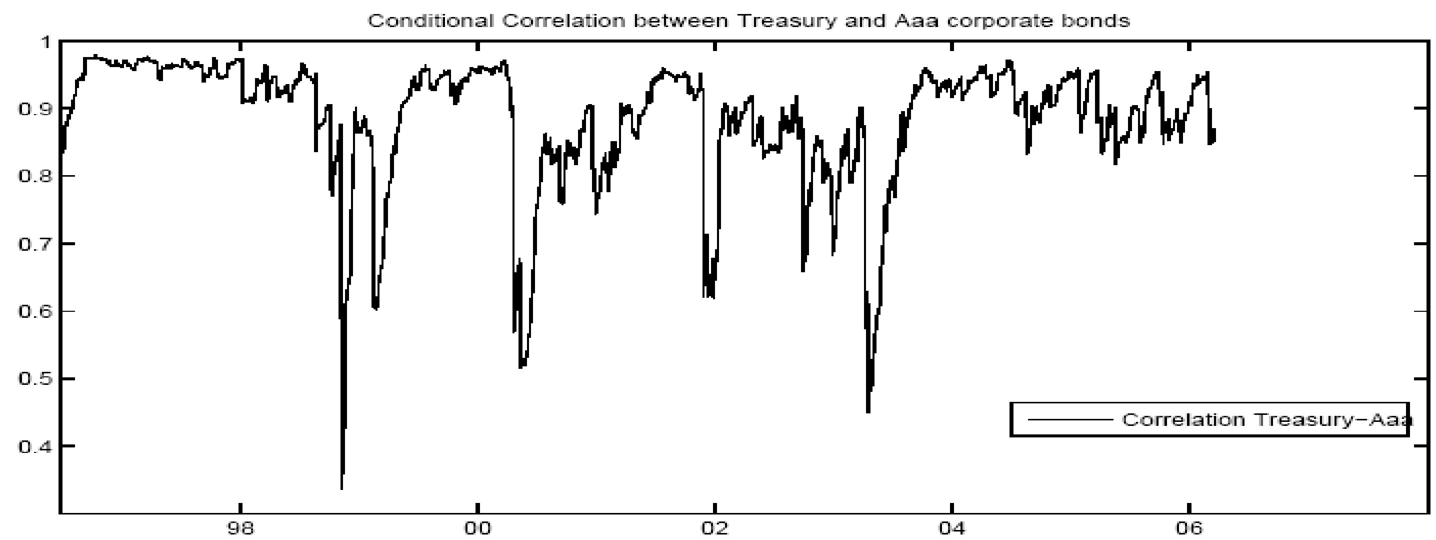
- The major goal is to diversify international equity risk
- We consider portfolios of US and DAX stock indeces (daily data from January 73 to December 2005)
- Horizon $T - t = 5$; Risk aversion $1 - \gamma = 3$.

CASE 2. Convergence trade (near-arbitrage) strategy starting from almost perfect correlation.

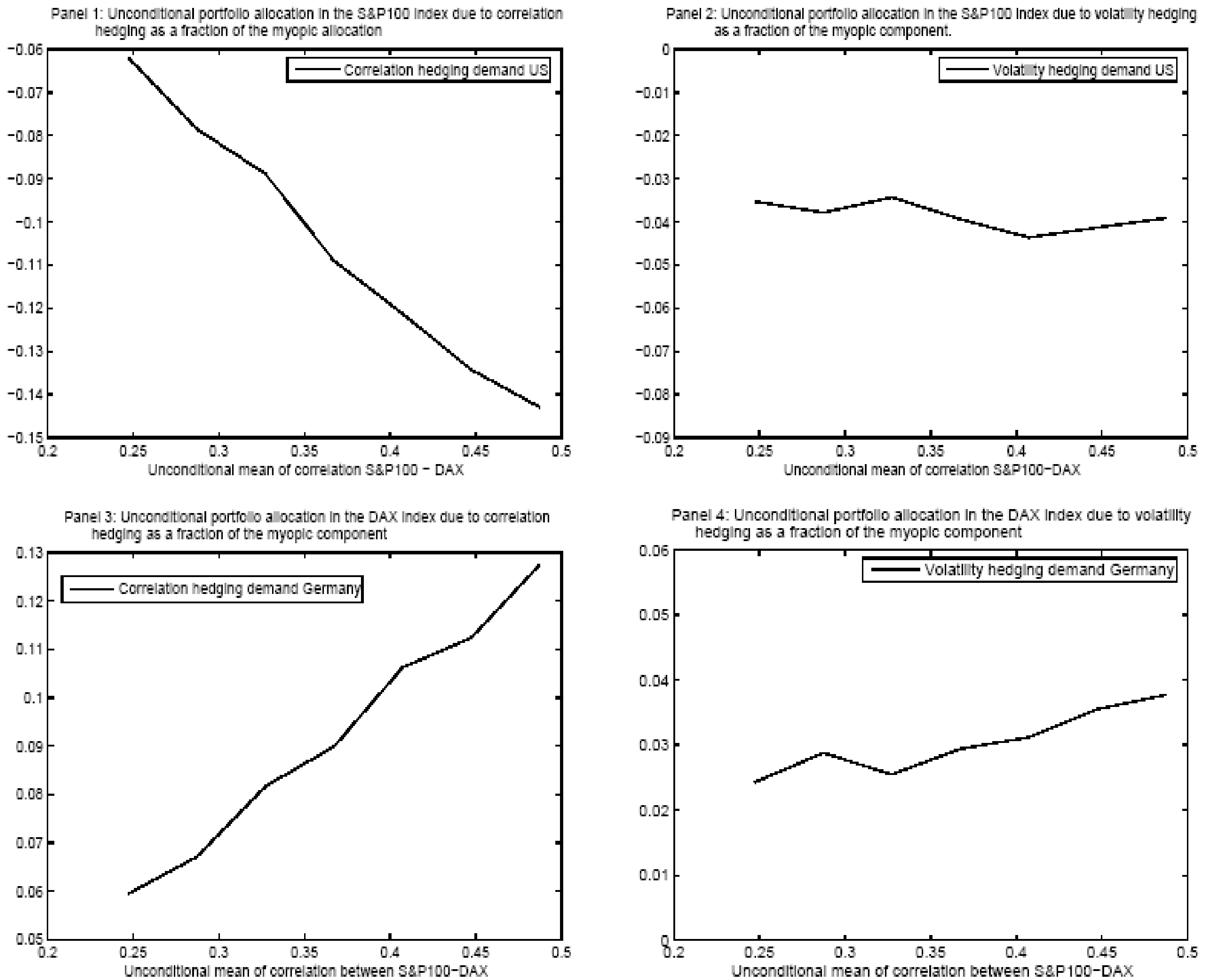
- The main goal is to hedge optimally the risk of a leveraged position in one asset, using a short position in another asset
- Consider 10 years (Treasury Bills - Aaa corporate) bonds pread (monthly data from January 83 to December 2005)

Estimated Correlation Processes

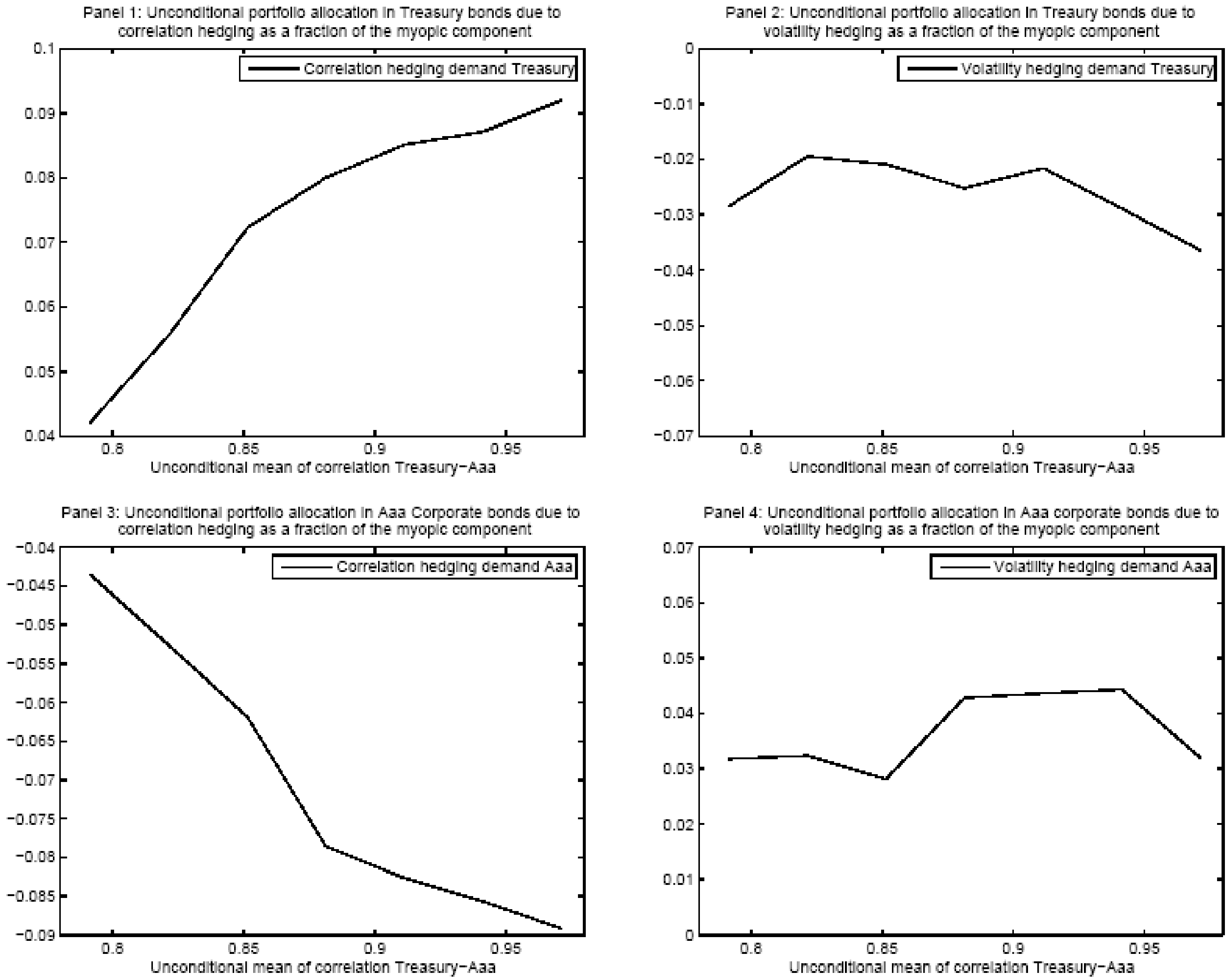




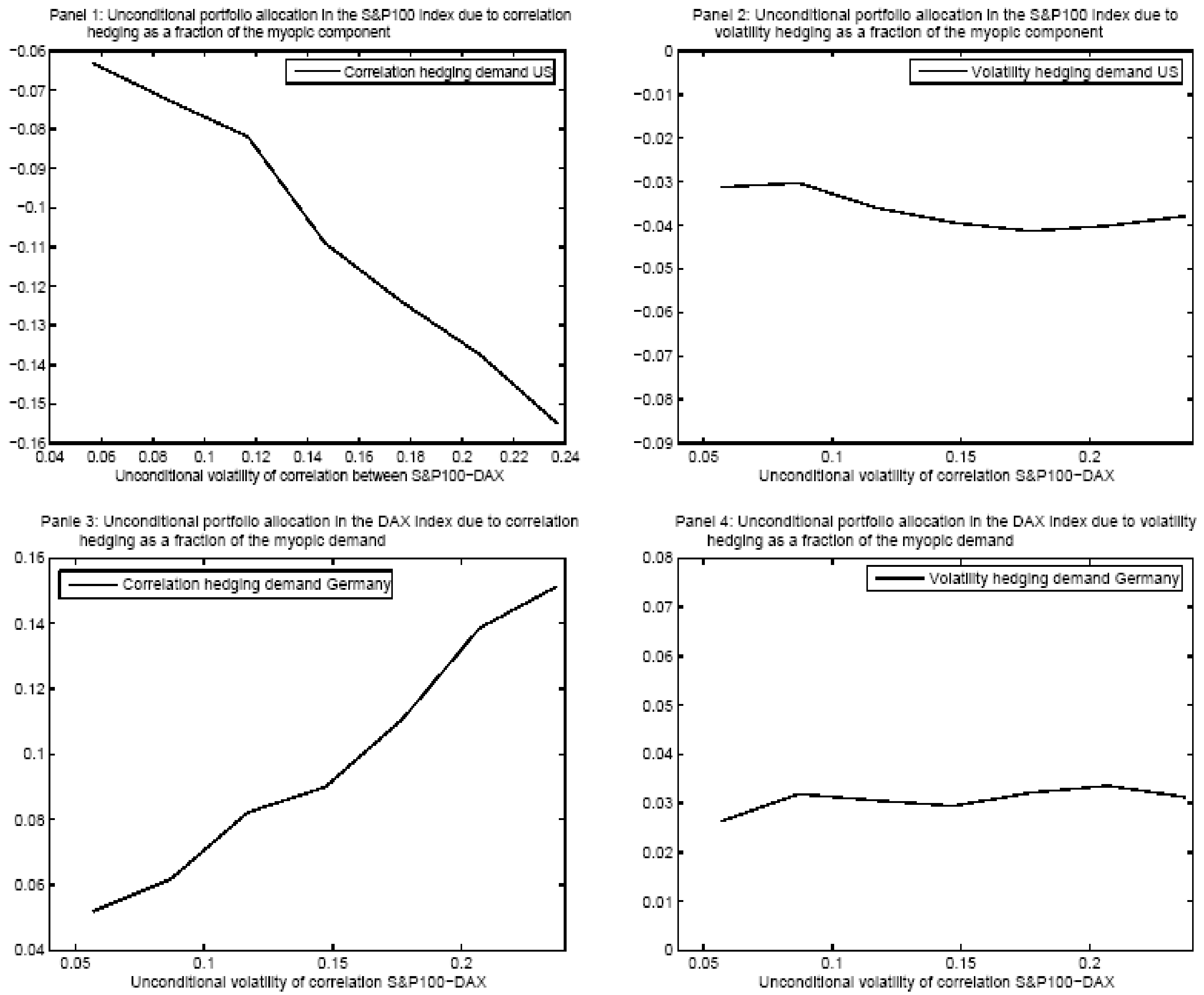
Stocks


Panel 1: Unconditional portfolio allocation in the S&P100 index due to correlation hedging as a fraction of the myopic allocation
Correlation hedging demand US
Unconditional mean of correlation S&P100 − DAX
Panel 2: Unconditional portfolio allocation in the S&P100 index due to volatility hedging as a fraction of the myopic component.
Volatility hedging demand US
Unconditional mean of correlation S&P100−DAX
Panel 3: Unconditional portfolio allocation in the DAX index due to correlation hedging as a fraction of the myopic component
Correlation hedging demand Germany
Unconditional mean of correlation between S&P100−DAX
Panel 4: Unconditional portfolio allocation in the DAX index due to volatility hedging as a fraction of the myopic component
Volatility hedging demand Germany
Unconditional mean of correlation between S&P100−DAX

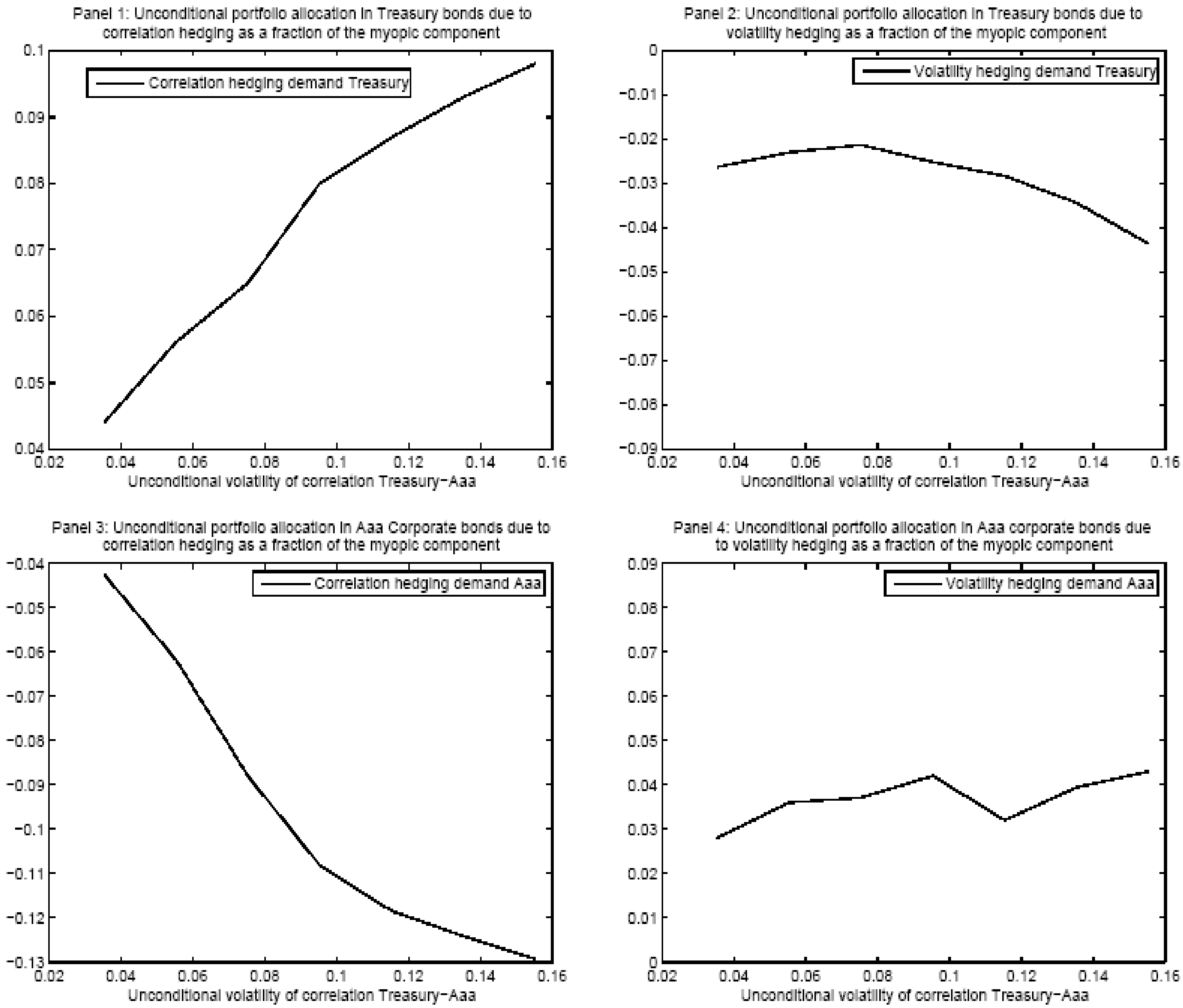
Bonds


Panel 1: Unconditional portfolio allocation in Treasury bonds due to correlation hedging as a fraction of the myopic component
Correlation hedging demand Treasury
Unconditional mean of correlation Treasury−Aaa
Panel 2: Unconditional portfolio allocation in Treaury bonds due to volatility hedging as a fraction of the myopic component
Volatility hedging demand Treasury
Unconditional mean of correlation Treasury−Aaa
Panel 3: Unconditional portfolio allocation in Aaa Corporate bonds due to correlation hedging as a fraction of the myopic component
Correlation hedging demand Aaa
Unconditional mean of correlation Treasury−Aaa
Panel 4: Unconditional portfolio allocation in Aaa corporate bonds due to volatility hedging as a fraction of the myopic component
Volatility hedging demand Aaa
Unconditional mean of correlation Treasury−Aaa

Stocks: The Volatility of Correlation



Bonds: The Volatility of Correlation



Volatility and correlation: very different dynamics!!

- Correlation shocks are much more persistent than volatility shocks! Correlations can deviate for several years from their average.

Panel A	mean of returns	volatility of returns	volatility of volatility
US	0.1106	0.1597	0.0574
Germany	0.1350	0.2013	0.0856
Treasuries	0.0165	0.0906	0.0107
Aaa	0.0184	0.0857	0.0100

Panel B	mean of correlation	volatility of correlation
US-Germany	0.3672	0.1469
Treasuries-Aaa	0.8841	0.0953

	Half-Life Equity		*Half-Life Bonds*
$\sigma(SP500)$	30 days	$\sigma(Treas)$	14 days
$\sigma(DAX)$	32 days	$\sigma(AAA)$	14 days
ρ	3 years	ρ	6 months

Investment Horizon

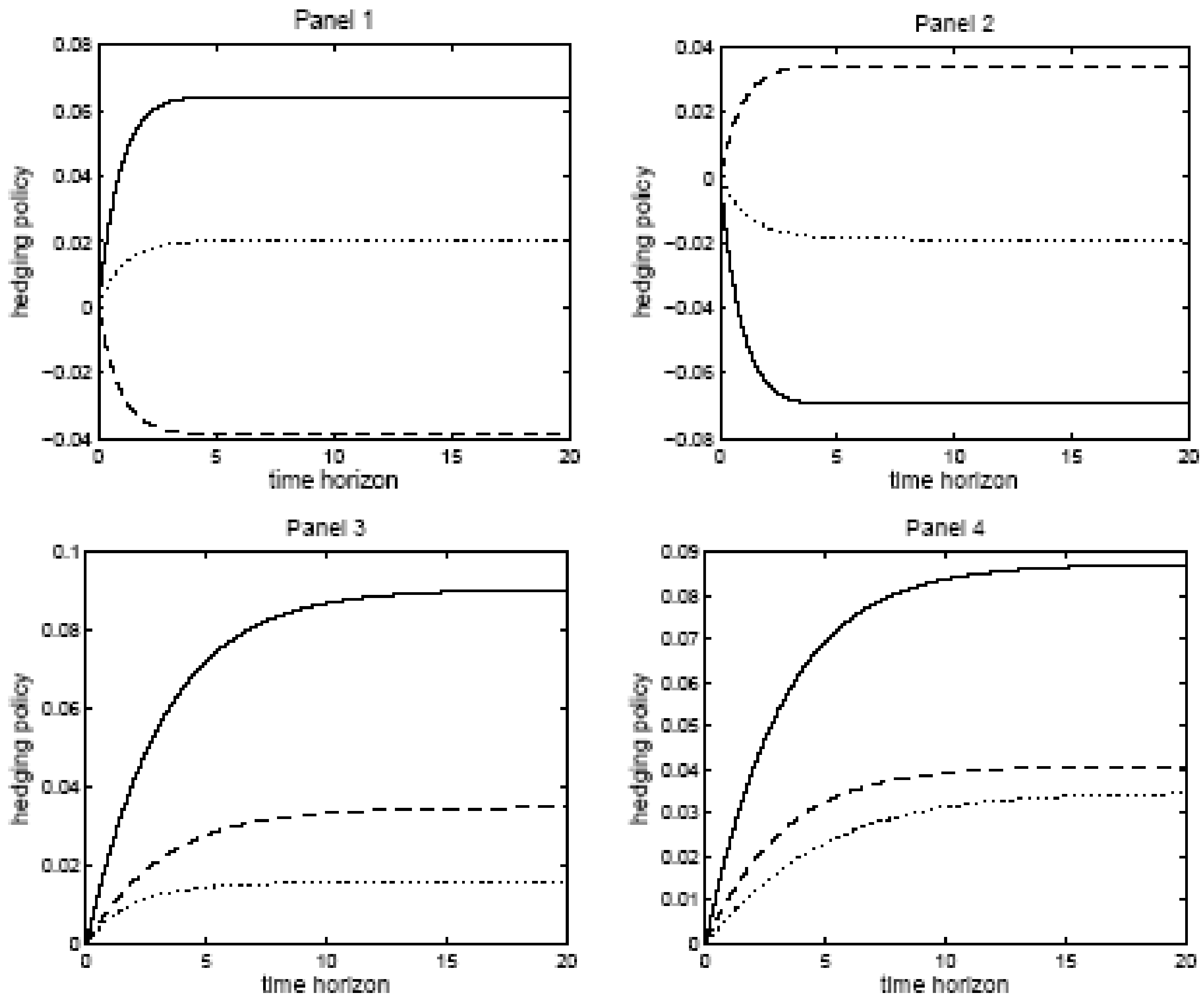
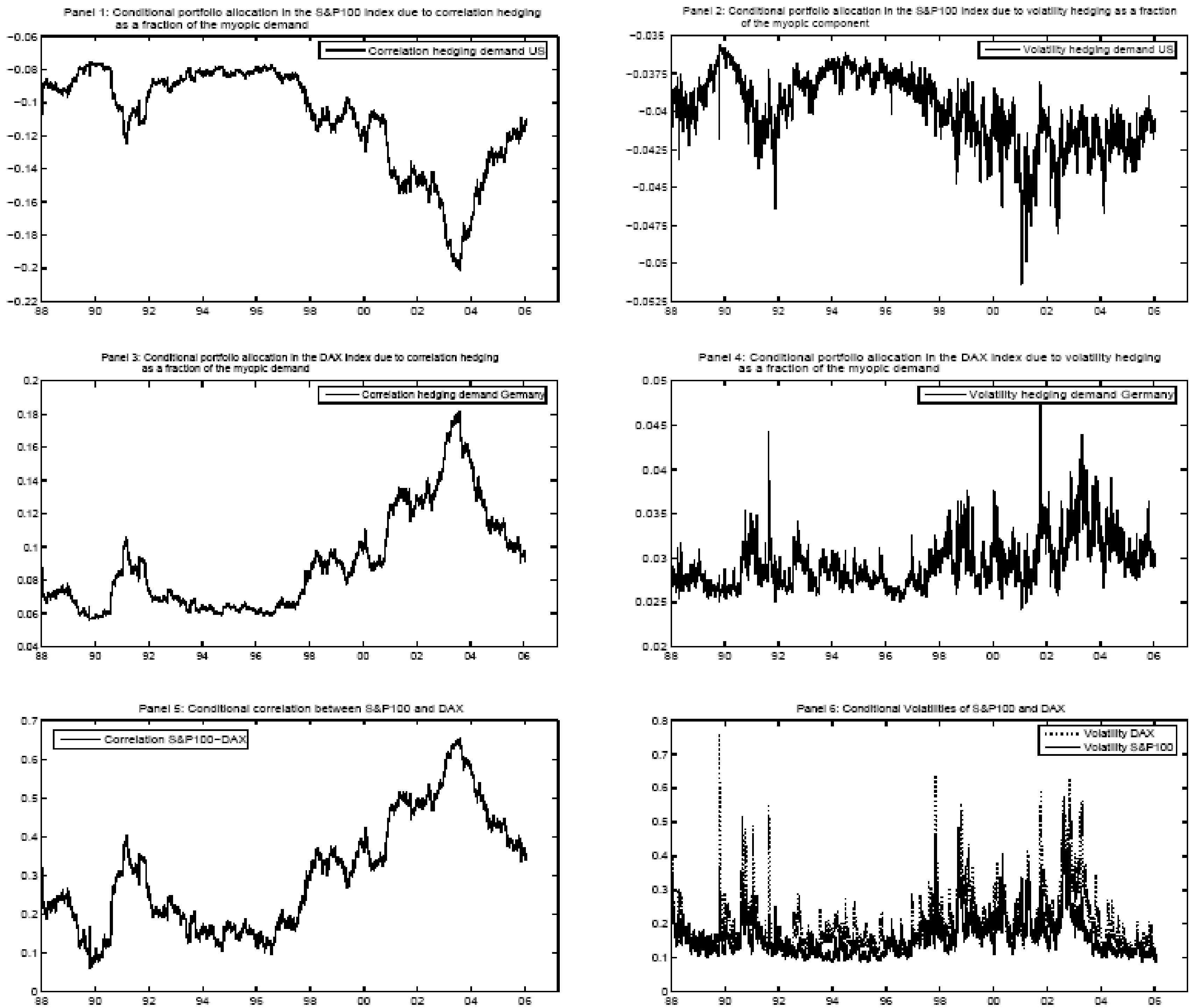
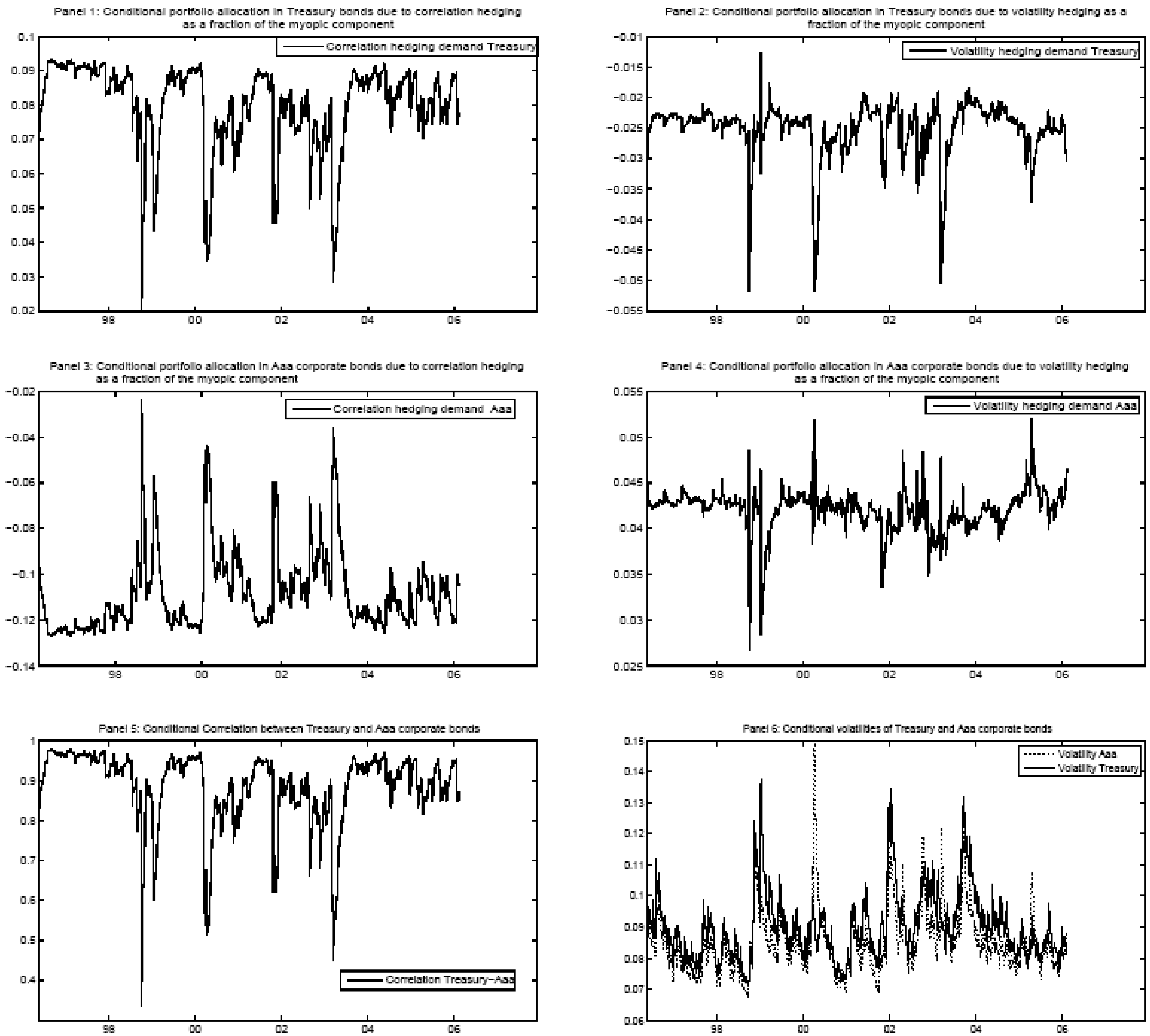


Figure 14. Horizon effect. Portfolio allocations due to correlation hedging (solid lines) and volatility hedging (pure volatility hedging, dotted lines, and hedging volatility influence on covariance risk, dashed lines). Policies are plotted as fractions of the corresponding absolute myopic allocations, for US Treasury bonds (Panels 1) and US Aaa corporate bonds (Panels 2) and for the S&P100 (Panels 3) and the DAX30 index (Panels 4). Time horizons are up to 20 years. The relative risk aversion coefficient used is $1 - \gamma = 3$.

Stocks: Time Variation



Bonds: Time Variation



Conclusions

- We solved in closed form the multivariate portfolio problem of an investor facing stochastic volatilities and stochastic correlations, modeled by a matrix-valued Wishart process.

- We incorporate in a tractable way several stylized features of asset returns as, e.g., volatility clustering, leverage effects and correlations mean reversion!

- In several calibrations, correlation hedging is economically relevant.

  1. Correlation hedging is larger than volatility hedging.
  2. Correlation hedging demand is quantitatively not negligible.
  3. Correlation hedging demand can be highly time-varying .
  4. Indirect utilities are very sensitive to correlation changes, and less to volatility changes!

Predictability

- Large literature showing that expected returns are time-varying and predictable by scaled price measures (Campbell and Shiller (1988), Fama and French (1989): dividend-price ratio)

- What are the consequences of predictability in returns for portfolio choice? (Brennan, Schwartz, and Lagnado (1997), Campbell and Viceira (1999), Kim and Omberg (1996), Barberis (2000), Balduzzi and Lynch (1999), Brandt (1999).. etc)

- We can extend our setup to introduce state-depended expected return:

$$\mathbb{E}_t\left(\frac{dS_1(t)}{S_1(t)}\right) = r + \alpha_1 \frac{S_1(t)}{D_1(t)}$$

- Then

$$\frac{S_i(t)}{D_i(t)} = tr(R_i Y(t))$$

where Y is an extended version of the original Σ

$$dY(t) = [\Omega\Omega' + MY(t) + Y(t)M']dt + \Sigma^{1/2}(t)dB(t)Q + Q'dB(t)'\Sigma^{1/2}(t)$$

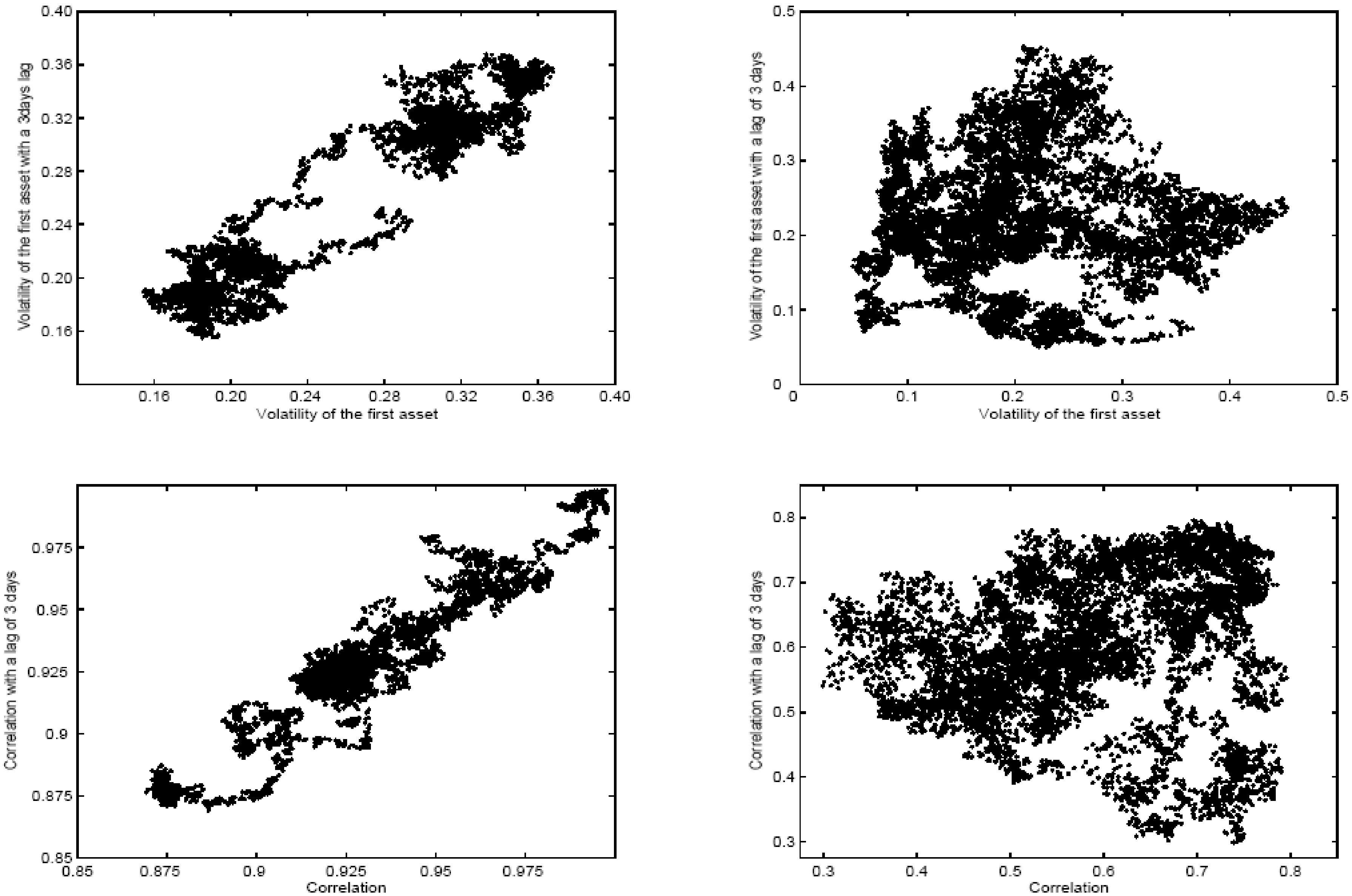
Thus the model becomes:

$$dS(t) = I_S\left[\begin{pmatrix} r + \alpha_1 tr(R_1 Y(t)) \\ r + \alpha_2 tr(R_2 Y(t)) \end{pmatrix} dt + \Sigma^{1/2}(t)dW(t)\right]$$

- Σ is either a block of Y, or more generally as $\Sigma = SYS'$, where S is a 2×3.

- The previous solutions apply.

Example 1: Volatility and Correlation Persistence



Example 2: Volatility and Correlation Leverage