Correlation Risk and Optimal Portfolio Choice*

ANDREA BURASCHI, PAOLO PORCHIA, and FABIO TROJANI*

ABSTRACT

We solve the intertemporal portfolio problem with correlation risk with a new approach modeling both volatility and correlation risk. The results show that correlation hedging is significant both qualitatively and quantitatively. We calibrate the model to empirical data and show that the portfolio hedging component for correlation risk can be as large as about 16% of Merton's myopic portfolio, whereas pure volatility hedging is on average less than 3%. The portfolio hedging component for correlation risk can be highly time varying and is larger in settings with high average correlations and correlation variances.

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THIS PAPER INVESTIGATES OPTIMAL INTERTEMPORAL PORTFOLIO DECISIONS in the presence of correlation risk. We study an economy in which the investment opportunity set is stochastic because of changes in both volatilities and correlations. In such an economy, the investor has a separate hedging demand for correlation risk. Depending on the economic scenario, we show that optimal portfolios can look very different from those obtained in a more common economic setting in which the investment opportunity set is affected only by time-varying expected returns and/or volatilities.

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The typical investment strategies pursued by defined benefit (DB) pension funds bear a striking resemblance to those used in long-short hedge funds. So why does this happen, and does it matter? A typical DB fund has a set of liabilities that behave like a mixture of inflation-linked and nominal bonds. The assets it will typically hold, however, rarely consist entirely of bonds designed to match the underlying liability. Most often, assets consist of some combination of equities and a broad range of actively managed bonds. In other words, the fund has a "short" position in long-dated real bonds and a "long" position in a set of other assets. During the second half of the 1990s, these strategies paid off and fund surpluses grew substantially. However, the subsequent bear market caused an equally sudden collapse in funding ratios so that in the space of just a few years it was common-place to see funding ratios swing in the United States from 115% of liabilities to just 80%. Current negative funding ratios have created the incentive for many investment managers to look even more aggressively for investment opportunities with higher expected returns. In 2005, pension funds globally invested more than \$77Bl of new money in alternative assets, up from \$62Bl in 2004. Property took the biggest share of this new money, at 35%, followed by 34% in funds of hedge funds. Retirement funds are the single largest holder of alternative assets, accounting for about \$465Bl, or 37%, of the total assets covered by the survey. Some of the most popular investment strategies of hedge funds are so called market-neutral and alpha-positive leveraged strategies. These consist of a large long position in one asset funded by an equally short position in one with (hopefully) similar risk characteristics. Often these strategies promise large expected long-term returns as well as a positive short-term carry. However, they hide an important risk: correlation risk.

To appreciate the portfolio impact of correlation risk, let us consider the case of a fund with \$1Bl of assets trading in the late 1990s. Assume that the fund uses an unconstrained mean-variance portfolio strategy and historical data (one year rolling windows with data sampled at daily frequency) to construct optimal portfolios. In the period between April 1996 and the beginning of August 1998, the implied correlation coefficient between the 10-year US Treasury bond yield and the Aaa corporate bond yield ranged between 0.95 and 0.99, as highlighted in Figure 1.

Insert Figure 1 about here.

Then, on August 17, Russia defaulted on its sovereign debt. Credit and liquidity spreads increased in all markets and the correlation between US Treasury and Aaa corporate bond yields rapidly declined to 0.80 in May 1999. At the beginning of August 1998, the average yield (standard deviation) of the two asset classes were 6.56% (0.65) and 7.24% (0.49), respectively. Based on historical observations and assuming a risk free rate equal to 5.2%, the tangency portfolio would have been given by a \$3.44Bl corporate bonds long position and a \$2.44Bl Treasury bonds short position. The expected portfolio return and standard deviation are 10.118% and 0.455, respectively. In May 1999, after the drop in the correlation coefficient, the new tangency portfolio would have been composed of a \$1.6Bl long position in corporate bonds and a \$0.6Bl short position in Treasury bonds. The change in the correlation coefficient alone would have induced a large portfolio reallocation and the hedge fund would have had to liquidate \$1.84Bl of corporate bonds. The reallocation effect would have been even larger if the hedge fund had operated subject to a Value-at-Risk (VaR) target. Let us assume a VaR target of \$1.47Bl (at a confidence level of 95% over a horizon of 90 days). In this case, the long position in corporate bonds (which is financed with a short position in Treasury bonds and borrowing at 5.2%) would change from \$5.32Bl to \$2.84Bl. The initial position in corporate bonds is higher since the overall optimal leverage before the change in correlation is higher. Under this constraint, however, the drop in the correlation coefficient would have a larger effect on the optimal portfolio and the fund would have to liquidate \$2.48Bl of the initial corporate bond position. Clearly, in an intertemporal setting these large portfolio reallocations are not ex-ante optimal. The expected portfolio return changes from 24.2% to 12.2%. If the fund manager had anticipated that the correlation coefficient between the two asset classes were not constant but stochastic, the optimal portfolio would have included a position to hedge changes in marginal utility due to unexpected changes correlations. This paper investigates this issue.

Time-varying correlation can become an important source of risk with wide ranging economic implications. Moskovitz (2003), for instance, argues that some pricing anomalies such as momentum and size effect can be explained by stochastic correlation. Driessen and Maenhout (2003) investigate the difference between the shape of the implied volatility smile in index and individual stock options. They document that the smile of individual stock options is flatter and argue that the reason for the substantial difference in the slope of the smiles is that correlation risk is priced and directly affects the value of index options.

In practical applications, modeling stochastic correlation is generating a vast interest because of the large synthetic Collateralized Debt Obligations (CDO) market. These structures consist of pools of Credit Default Swaps (CDSs) where tranching allows the creation of flexible default risk profiles. Since most of these new products involve a portfolio of firms, the time-variation of the correlations is a primary source of pricing and risk management issues. The list of financial instruments, where the value depends directly on the properties of the correlation process, is broad to the extent that they are now referred to as a separate asset class: "correlation derivatives." In this class, we find contracts designed to generate exposure to a foreign financial index (either a foreign interest rate or a stock market index) but with a payoff denominated in the domestic currency. The pricing, hedging, and risk management of these instruments explicitly depend on the correlation between the foreign index and the exchange rate. Examples include foreign-exchange quanto futures and options such as the Nikkei derivatives traded on the CME. Additional examples of correlation derivatives are differential swaps, basket options, and options written on the maximum or minimum of a set of assets.

A vast amount of literature has explored the implications of stochastic volatility for portfolio choice. Very little, however, is known about the impact of stochastic correlations. In part, the reason is that a sensible specification of a stochastic correlation process implies tight non-linear restrictions and boundary conditions on the asset return process. Correlations need to be bounded between -1 and $+1$, and the covariance matrix of asset returns needs to be symmetric and positive definite, so that a model with stochastic correlations can easily become analytically intractable. In this paper, we specify the covariance matrix as a matrix-valued affine diffusion process. This makes the model very tractable and allows for a natural interpretation of the solutions of the intertemporal optimal portfolio problem in a general multivariate setting. The transition probabilities of the diffusion process follow a Wishart distribution, which has been extensively studied in the discrete-time Bayesian literature to model priors on second moments. The optimal portfolio implications, however, are not known.

When correlations are stochastic, optimal portfolio weights include a hedging demand component for correlation risk. We use the model solutions to address the following questions:

(a) What is the economic importance of correlation risk in optimal portfolio choice? We calibrate the model and find that this hedging demand can account for as much as 16% of Merton's myopic portfolio. In contrast to volatility risk, correlation risk has a first order impact on optimal portfolio weights. This impact is larger for high average correlations and for large correlation variances. Extreme (positive or negative) correlations act as de-facto implicit portfolio constraints, since they limit the number of linearly independent investment opportunities. The extreme case is when asset correlations are equal to one, in which case markets attain the highest degree of incompleteness. In other words, stochastic correlations shift the degree of market incompleteness, affecting the possibility to finance the optimal consumption plan of the agent. The likelihood of not replicating adequately the selected terminal wealth level changes the optimal portfolio that would otherwise arise under a constant correlation. This effect is increasing in the investment horizon and decreasing in the speed of mean reversion of the correlation process, which determines the unconditional variance of correlations.

(b) What is the extent of time variation of the optimal hedging demand for correlation risk? When the price of correlation risk is not constant, the hedging demand for correlation risk is state dependent. The time variation is the highest during periods of market dislocation, e.g. during the fall of 1998. We calibrate the model and find that in these periods correlation risk is the most important source of variation in portfolio weights. Therefore, the results about the importance of intertemporal hedging in a multivariate setting with stochastic correlations are very different from those obtained in the scalar case with stochastic volatilities.

This paper draws from the contribution of a large literature on optimal portfolio choice under a stochastic investment opportunity set. A first set of papers studies optimal portfolio and consumption problems when the instruments consist of one riskless and one risky asset.[1] Kim and Omberg (1996) solve the portfolio problem of an investor optimizing utility of terminal wealth, where the riskless rate is constant and the risky asset has a mean reverting Sharpe ratio and constant volatility. Wachter (2002) extends this setting to allow for intermediate consumption and derives closed-form solutions in a complete markets setting. She also shows that in this setting the optimal portfolio is a weighted average of the optimal portfolios maximizing a sequence of utility of terminal wealth. Ferretti and Trojani (2004) use asymptotic methods to study the incomplete markets case. Chacko and Viceira (2005) consider an infinite horizon economy with Epstein-Zin preferences where the risky asset volatility follows a mean reverting square-root process. They characterize, in closed form, the optimal hedging demand in the case of unit elasticity of inter-temporal substitution of consumption, and obtain approximate solutions in all other cases. We contribute to this literature in three main respects. First, with respect to the first set of papers, we extend the analysis to allow for stochastic volatility. Second, we consider an economy with multivariate risk factors in which the correlation between risk factors is stochastic. Third, we investigate the optimal portfolio implications when markets are incomplete. Clearly, this is important when volatilities and correlations are stochastic since, in this case, it is problematic to invert the relationship between asset prices and the state vector to improve the spanning of the state-space. We identify and study the optimal portfolio choice implications of correlation risk by modelling directly a stochastic process for the covariance matrix that automatically satisfies the necessary regularity conditions. In order to derive closed-form solutions, we assume CRRA preferences. This is more restrictive than Chacko and Viceira (2005), who consider a more general set of Epstein-Zin preferences that allows them to disentangle risk aversion and intertemporal substitution effects on optimal portfolios. Even in their case, however, closed-form solutions are available only in the special case of a unit elasticity of intertemporal substitution and an infinite horizon investor. The last assumption allows them to focus on the steady state solution. We differ from their setup by providing closed-form solutions for the finite horizon problem.

Multivariate portfolio selection problems have been considered in a further series of papers. The majority of those, however, assumes that volatilities and correlations are constant. In such settings, no intertemporal hedging against stochastically changing covariance matrices arises. Brennan and Xia (2002) analyze optimal asset allocations under inflation risk, when the investor can use cash, a stock, and a nominal bond as investment vehicles. They assume that expected inflation and real rates follow Ornstein-Uhlenbeck processes and that the real pricing kernel and the nominal stock price return have constant volatilities. Sangvinatsos and Wachter (2005) study the portfolio problem of a long-run investor when both nominal bonds and stocks are available for investment and the term structure is Gaussian. Mulk and Sorensen (2004) propose a multivariate setting for optimal bond portfolio selection, which can account for non-Markovian interest rate dynamics in a Heath, Jarrow, and Morton type of model. In a complete market and for non-Markovian Gaussian dynamics, they provide closed-form optimal consumption and investment solutions. The main difference between our setting and these approaches is that we focus on optimal portfolio demand under stochastic volatilities and correlations of asset returns. Liu (2005) derives in closed-form optimal portfolios for an economy in which the admissible instruments for investment include a risk-less asset, a bond and a stock. Bond returns are described by the Cox, Ingersoll, and Ross' (1985) model, and stock returns display stochastic volatility. To obtain closed-form portfolio policies he assumes independence between the state variables driving pure term structure risk and the additional risk factors influencing stock return volatility. Covariances and correlations in his model are stochastic, but only to the extent that they are functions of stock and bond returns volatility. Therefore, variances and covariances are linked by a deterministic relation, and optimal hedging portfolios do not allow for a separate role of stochastic variance and correlation risk. In contrast to Liu (2005), we focus on the implications of stochastic correlation risk on optimal portfolio choice in relatively general stochastic process for the correlation structure of the risk factors. In such a setting, our closed-form solutions hold for an arbitrary dimension of the universe of assets.

The article is also related to the work of Das and Uppal (2005) who investigate the extent to which systemic risk implied by the arrival of common Poisson shocks affect the gains to international diversification. When the arrival of these rare events are common across assets, stock returns correlation changes thus affecting the optimal portfolio allocation. The main difference with respect to their work is that we investigate a model in which correlation shocks can be persistent and therefore may induce larger portfolio rebalancing decisions. Liu, Longstaff, and Pan (2003) consider the case in which a financial event may affect both market prices and volatility using the double-jump framework in Duffie, Pan, and Singleton (2000). They characterize the optimal portfolio choice and show that the optimal policy is similar to the one of an investor facing short-selling and borrowing constraints even if none are imposed. Although their approach allows for a rather general model with stochastic volatility, they focus on a single risky asset economy and do not model stochastic correlations.

The article is also related to the multivariate-Garch econometrics literature that studies dynamic specifications of the covariance matrix. Bollerslev, Engle, and Wooldridge (1988) propose a model in which each element of the covariance matrix is assumed to be a linear combination of the squared returns: $vech(\Sigma_t) = A\ vech(R_{t-1}R'_{t-1}) + B$. However, this model suffers from the curse of dimensionality[2] and it does not necessarily imply a positive definite covariance matrix. Bollerslev (1987) proposes a solution in which the correlations are restricted to be constant. To overcome this issue, Engle and Sheppard (2001) have suggested a specification with time-varying correlations compatible with the estimation of a set of univariate Garch processes. Alternatively, Harvey, Ruiz, and Sheppard (1994) model Σ_t as a function of a set of stochastic factors h_t using the decomposition $\Sigma_t = A(h_t \times I)A'$ assuming that the vector of factors h_t follows a Gaussian VAR process. However, in both cases, the correlation dynamics are driven by the same factors affecting the volatility. There is no independent role for the correlation process. An important difference of our approach is that the correlation process can have independent dynamics from the factors driving the volatility process.

In Section I, we summarize the empirical properties of the correlation between different sets of assets and the potential implications for portfolio choice. Section II introduces the basic model specification, highlights the main properties of the implied correlation process and presents the solution to the portfolio problem under stochastic covariance matrices. In Section III, we calibrate the model to empirical data and quantify the impact of stochastic correlation risk on intertemporal hedging. In Section IV, we discuss some extensions of the model. We consider economies in which the covariance process is not autonomous, namely it depends on other economic variables, such as the level of interest rates or general market conditions, beyond the covariance itself. This specification can describe settings in which the speed of mean reversion and/or the volatility of correlations is state dependent. In this economy, we also relax the assumption of constant interest rates. Section V investigates the implications of correlation risk in the case of a pension fund manager trying to hedge inflation risk using both index-linked and nominal Treasury bonds. Section VI concludes. All proofs are in the Appendix.

I. Why Correlation Hedging?

Since Bollerslev, Engle and Wooldridge (1988) and Bollerslev, Chou and Kroner (1992) an increasing body of empirical evidence has documented that several asset returns exhibit time-varying variances and covariances. Time variation in covariances can derive from both changing volatilities and changing correlations. Empirical evidence supporting time-varying correlations includes several contributions. For instance, Longin and Solnik (1995) find that correlation increases in periods of high volatility. Ledoit, Santa-Clara and Wolf (2003) study the comovement of international stock markets and find that the level of correlation depends on the phase of the business cycle. Moreover, Erb, Harvey and Viskanta (1994) find that international markets tend to be more correlated when countries are simultaneously in a recessional state. Moskowitz (2003) studies the link between the cross-section of stock expected returns and the time-variation in the conditional covariance of portfolios of US assets. He finds that covariances across portfolio returns are highly correlated with NBER recessions and that average correlations change substantially over time. Similar evidence is found by Ang and Chen (2001) who show that the correlation between US stocks and the aggregate US market is much higher during extreme downside movements, than during upside movements. These different behaviors of correlations over time and states have motivated the theoretical literature to study models in which some of these features arise endogenously in a rational expectations equilibrium. Ribeiro and Veronesi (2002) propose a model where excess stock comovement during bad times is obtained endogenously as a reflection of higher uncertainty. When estimated with data on seven major countries, their model can replicate the historical patterns of international average correlations.

To appreciate the time variation in the correlations of some asset returns, Figure 2 plots the estimated conditional correlations based on Engle's (2002) Dynamic Conditional Correlations (DCC) model for US and German equity index daily returns (top panel) and 10 years Treasury Bill and Aaa-corporate bond index daily returns (bottom panel), in the periods 1988–2005 and 1996–2006, respectively.

Insert Figure 2 about here.

In both cases, the estimated parameters in the DCC correlation dynamics are significant at standard significance levels, highlighting a persistence in correlation changes. In the top panel of Figure 2, the unconditional correlation between US and German index returns is about 0.37. The extent of time variation, however, is large, ranging between 0.1 in the late eighties and more than 0.6 towards the end of the sample. Although correlations tend to spike in times of extreme market distress, they are generally quite persistent.

In the bottom panel, the average correlation of treasury Bills and Aaa-bond yields is about 0.88. Before 1998, bond correlations were very stable and even higher, most of the time, than their unconditional mean. After the 1998 Russian crisis, correlations virtually collapse to a lowest value of less than 0.4 and become highly time-varying, to recover to a level near to their unconditional value only towards the end of 2003. Bond correlations are generally quite persistent, with occasional spikes during extreme market distress, as in the most recent international financial crises.

The empirical analysis confirms what suggested by a visual inspection: the time series properties of correlation are very different from the properties of volatility. First, the autocorrelation of the correlation is larger and decays slower than the one of the volatility. Thus, correlation shocks are much more persistent than volatility shocks. For instance, the estimated half life of the equity index volatility is about 30 and 32 days for the US and the German market, whereas the estimated half life of the correlation between US and German equity index returns is about three years. Similarly, the estimated half life of Treasury and corporate bonds volatilities is less than fourteen days, whereas the estimated half-life of their correlation is about half a year.

Second, the volatility changes quite quickly in response to market shocks and tends to cluster in periods of high (low) volatility. Figure 3 illustrates this point by plotting the estimated DCC volatility patterns of US and UK equity index returns (top panel) and those of Treasury Bill and Aaa-corporate bond index returns (bottom panel).

Insert Figure 3 about here.

Volatility changes can be quite large. For instance, US and German equity volatilities range from about 10% to about 75%, where the highest volatilities were realized around 1990 and 1998. The variability in Treasury bills and Aaa corporate bond volatilities is lower than for stocks, but is still substantial: estimated conditional volatilities for bond returns range from a minimum of about 7-8% to a maximum of about 15%-16%.

The estimated correlations in Figure 2 can deviate for several years from their unconditional mean, whereas clusters of high or low volatilities in Figure 3 typically last for a much shorter period of time. This evidence suggests that correlation risk can be potentially important for optimal portfolio choice and very different with respect to the implications due to stochastic volatility, which several authors have found to be small. Chacko and Viceira (2005), for instance, find that volatility hedging amounts to less than 4% of Merton's myopic portfolio, for realistic levels of risk aversion and elasticity of intertemporal substitution.

II. The Model

An investor with CRRA utility over terminal wealth trades 3 assets, a riskless asset with instantaneous riskless return r, and two risky assets, in a continuous-time frictionless economy on a finite time horizon $[0, T]$. Our analysis extends to opportunity sets consisting of any number of risky assets and correlation factors, without affecting the existence of closed-form solutions and their general structure. We focus on the 2-dimensional setting, in order to preserve the key economic intuition about optimal portfolio choice with stochastic correlations from notational complications.

A. Portfolio Allocation Problem

The investment opportunity set can be stochastic both because of changes in expected returns and conditional variance and covariances. It is well known that in order to obtain tractable solutions one needs to impose restrictions on the functional form of the squared Sharpe-ratio, or of the maximal squared Sharpe-ratio in incomplete markets. For instance, in standard settings affine maximal squared Sharpe-ratios imply affine solutions. Given an affine state variable process, two options are available. The first is to assume constant expected returns and an affine inverse covariance process; the second one is to assume time varying expected returns with an affine covariance process and a constant price of risk λ (see, e.g., Chacko and Viceira (2005) and Liu (2006)). In this article we investigate the second setting, since it allows an easier interpretation of the dynamic properties of the model in our multivariate context. The cum dividend evolution of the price vector $S = (S_1, S_2)'$ of the risky opportunities is described by the bivariate stochastic differential equation:

$$dS(t) = I_S \left[(r\bar{1}_2 + \lambda(\Sigma, t))dt + \Sigma^{1/2}(t)\, dW(t) \right] \quad ; \quad I_S = \text{Diag}[S_1, S_2] \tag{1}$$

where $r \in \mathbb{R}^+$, $\bar{1}_2 = (1,1)'$, W is a standard two-dimensional Brownian motion and $\Sigma^{1/2}$ is the positive square root of returns conditional covariance matrix Σ. The available investment opportunity set is stochastic because of the time varying risk premium $\lambda(\Sigma, t)$ which is a function of the stochastic covariance matrix Σ. The diffusion process for Σ is detailed below. The risk premia on the two assets are time-varying and state dependent. Let $\pi(t) = (\pi_1(t), \pi_2(t))'$ denote the proportion of wealth $X(t)$ invested in the first and the second risky asset. Agent's wealth evolves as

$$dX(t) = X(t) \left[r + \pi(t)'\lambda(\Sigma, t) \right] dt + X(t)\pi(t)'\Sigma^{1/2}(t)dW(t) \tag{2}$$

The agent selects the portfolio process π, in order to maximize the CRRA utility of terminal wealth with RRA coefficient $1-\gamma$. If $X_0 = X(0)$ denotes the initial wealth, and $\Sigma_0 = \Sigma(0)$ denotes the initial covariance matrix, the investor's optimization problem is:

$$J(X_0, \Sigma_0) = \sup_{\pi} \mathbb{E}\left[\frac{X(T)^\gamma - 1}{\gamma} \right], \tag{3}$$

subject to the dynamic budget constraint (2).

B. The Covariance Matrix Process

To model stochastic covariance matrices in a convenient way, we make use of the continuous-time process introduced in Bru (1991) and studied by Gourieroux and Sufana (2003) and Gourieroux, Jasiak and Sufana (2004). This diffusion process is a matrix-valued extension of the univariate square-root process that gained popularity in the term structure and the stochastic volatility literature; see, e.g., Cox, Ingersoll and Ross (1985) and Heston (1993).

Let Z be a bivariate standard Brownian motion independent of W and define $B(t) = [W(t)\ Z(t)]$ as a 2×2 matrix-valued standard Brownian motion. The diffusion process for Σ is defined as:

$$d\Sigma(t) = \left[\Omega\Omega' + M\Sigma(t) + \Sigma(t)M' \right] dt + \Sigma^{1/2}(t)dB(t)Q + Q'dB(t)'\Sigma^{1/2}(t) \tag{4}$$

with Ω, M, Q being 2×2 square matrices.

This process satisfies five important properties that make it ideal to model stochastic correlation. First, it implies that if $\Omega\Omega' \geq QQ'$ for some $k \geq 1$ then $\Sigma(t)$ is positive definite and follows a Wishart distribution; see Bru (1991). Under this condition, the implied correlation process is well behaved and bounded between -1 and $+1$. Second, the process (4) is affine in the sense of Duffie and Kan (1996) and Duffie, Filipovic and Shachermayer (2003). This feature implies closed-form expressions for all conditional Laplace transforms. Third, it is possible to show that the conditional distribution for $\Sigma(t)$ is Wishart. This distribution has been studied in Bayesian statistics to model priors on multivariate second moments but it has never been used to study optimal portfolio choice. Four, if $d\ln S_t$ is a vector of returns with Wishart variance Σ_t, then the variance of a portfolio π is a Wishart process. This is generally not the case for Garch models as they are not invariant under linear aggregation. Fifth, the model is flexible enough to fit many of the empirical features of financial assets returns, such as leverage and co-leverage which are found to be important empirical properties in the literature.

The only thing left to specify is the risk premium $\lambda(\Sigma, t)$ as a function of the state variable. To motivate a choice for λ, we notice that under power utility function, in a Breeden's (1979) consumption-based model, the price of volatility risk is equal to $\gamma Cov\left[dC/C, dS/S\right]$ where $C(t)$ is the aggregate consumption. If aggregate consumption follows $dC/C = \mu_c(C_t)dt + a'\Sigma^{1/2}dB(t)b$, where a and b are two $n \times 1$ vectors, then the risk premium is given by $(\gamma - 1)Cov\left[dC/C, dS/S\right] = (\gamma - 1)\left[a'\Sigma^{1/2}dB(t)b, e_i'\Sigma^{1/2}dB(t)e_1\right]$. Using the property that $Cov\left(dB(t)a, dB(t)b\right) = a'bIdt$, it is easy to show that the risk premium is affine in $\Sigma(t)$. Clearly, the result is more general and holds in any economy whose stochastic discount factor has local volatility equal to $a'\Sigma^{1/2}dW(t)b$. Thus, we consider economies in which the vector of market prices of risk is linear in $\Sigma(t)$

$$\lambda(\Sigma, t) = \Sigma\lambda$$

where $\lambda = (\lambda_1, \lambda_2)' \in \mathbb{R}^2$. The same assumption is made in Heston (1993) in the case of a scalar economy. To study the properties of the correlation process under a Wishart diffusion (4), we can use Itô's Lemma to compute the correlation dynamics.

Proposition 1 *Let ρ be the correlation diffusion process implied by covariance matrix dynamics (4). The instantaneous drift and conditional variance of $d\rho(t)$ are given by:*

$$E_t\left(d\rho(t)\right) = A(t)\rho(t)^2 + B(t)\rho(t) + C(t) \tag{5}$$

$$E_t\left(d\rho(t)^2\right) = \left(1 - \rho^2(t)\right)\left(E(t) + G(t)\rho(t)\right) \tag{6}$$

where coefficients A, B, C, E, G depend exclusively on Σ_{11}, Σ_{22} and the model parameters Ω, M and Q.[3]

Despite the affine structure of the covariance matrix process, the correlation dynamics is non-linear. By construction, the instantaneous returns correlation $\rho = \Sigma_{12}/\sqrt{\Sigma_{11}\Sigma_{22}}$ is bounded in the interval $[-1, 1]$: No explicit additional constraint on the correlation process is needed in order to ensure a well defined return covariance matrix process. The instantaneous drift and volatility of the correlation process are quadratic and cubic in ρ. The correlation dynamics is not autonomous: Both the drift and the instantaneous variance have coefficients that depend on the level of the volatilities of the first and the second asset return. Notice that the correlation process implied by an affine covariance matrix process is not affine. Moreover, it is different from the one in Driessen and Maenhout (2005), who assume a process with linear mean reverting drift and quadratic local volatility.

The matrix M drives the speed of mean reversion and the autocorrelation features of covariances and volatilities.[4] The matrix Q drives the covariance volatility and the volatility of volatility. The correlation drift and volatility implied by the covariance matrix process (4) is illustrated in Figure 4, where we make use of the parameters calibrated to the daily US and German equity index return series used to plot the top panel of Figure 2.[5]

Insert Figure 4 about here

The drift of the correlation is a flat function of ρ when the volatilities of asset one and two are similar. When the volatility of asset one is small (large) relative to the volatility of asset two, the correlation drift is negative (positive) for a broad set of values of ρ and takes the smallest (largest) values when the correlation is about zero. As expected, for perfect positive and negative correlations the drift is always negative and positive, respectively, for all volatility values. The correlation volatility is zero at the borders when $\rho = 1, -1$ and is highest for correlations around zero. The pattern of the correlation volatility function is quite similar for different levels of the volatilities of assets one and two.

To illustrate the flexibility of model (4) in generating rich and realistic correlation properties, Figure 5 presents two simulated realizations of the correlation process implied by parameter choices calibrated to the time series of asset returns in Figure 2.

Insert Figure 5 about here

The top panel of Figure 5 presents a realization of a correlation process similar to the one obtained in the top panel of Figure 2 for the estimated correlations of international equity index returns. The bottom panel of Figure 5, instead, presents a realization of a different correlation process, which is more similar to the correlation patterns estimated in the bottom panel of Figure 2 for Treasury and Aaa corporate bond index returns. The simulations of Figure 5 were obtained from a process with parameters calibrated to the unconditional moments of US and German stock index returns and Treasury and Aaa corporate bond index returns, respectively.

To appreciate the ability of the model to fit the most important features of the time series of correlations and volatilities, we also investigate the extent to which different parametrization of M and Q can reproduce different empirical properties of asset returns.

A. Persistence of the correlation and volatility. Figure 6 presents scatter plots of simulated daily return volatilities and correlations, for two different parameterizations of the Wishart process (4), plotted against their own past value three days before.

Insert Figure 6 about here

Panels on the left have been produced by a parameterization generating a positive autocorrelation in volatilities for the return of asset one, leading to the well-known volatility clustering phenomenon. This feature is reflected in the positive relation between future and past volatilities, emerging in the top left panel. For the same parameterization, the positive relation between future and past return correlations, in the left bottom panel, shows a persistence also in the correlation dynamics. Panels on the right, instead, have been obtained for a parameterization in which no substantial persistence in volatilities and correlations arises.

B. Volatility and correlation leverage. Black's volatility 'leverage' effect, that is the negative correlation between returns and volatility, has been often found an empirical feature of stock returns and it is explicitly modeled by Heston (1993) to reproduce the empirical regularities of option-implied volatility skews.[6] Roll's (1988) correlation 'leverage' effect, that is the negative covariance between returns and average correlation shocks across stocks, is also a feature supported by empirical evidence; see, e.g., Ang and Chen (2002). To understand the mechanism producing the volatility and correlation leverage effects in our model, it is important to notice that the return dynamics (1) and the Wishart dynamics (4) can be instantaneously correlated. This property induces a genuine hedging demand against correlation risk and is an important advantage with respect to, for instance, discrete-time multivariate GARCH-type models with dynamic correlations (see, e.g., Engle, 2002, Ledoit, Santa Clara and Wolf, 2003, Audrino and Trojani, 2004, and Pelletier, 2004), where volatilities and correlations are conditionally uncorrelated with asset returns.

In addition to controlling the returns volatility of volatility, the matrix Q in equation (4) controls the correlation between returns, volatility and correlation shocks. The parameters q_{11}, q_{12} in the first row of this matrix control the sign of the correlations between returns and variance-covariance shocks. The parameters in the second row determine more directly the level of this correlation and the returns volatility of volatility. When q_{11} and q_{12} are both negative, the model implies a volatility-leverage and a correlation-leverage effect. When both these parameters are zero, no leverage effect arises. Figure 7 presents some scatter plots of simulated returns, plotted against contemporaneous changes in volatilities and correlations.[7]

Insert Figure 7 about here

In the panels on the right, both a volatility and a correlation leverage effect for the return of asset one emerges, which is highlighted by the negative relation between returns and volatilities (top right panel) and between returns and correlations (bottom right panel). Panels on the left, instead, depict a setting in which no leverage effect arises.

Example 1. The structure of leverage effects implied by returns dynamics (1) and (4) is formally analogous to the one arising in the scalar setting with a stochastic volatility process satisfying a Heston-type model. This return dynamics has been studied, among others, in Liu (2001) and Chacko and Viceira (2004). In the univariate case, process (4) is Heston (1993) stochastic volatility model. Given a standard bivariate Brownian motion $B = [W \;\; Z]$, parameter $Q = [q_1 \; q_2]'$ can be chosen so that:

$$\frac{dS(t)}{S(t)} = (r + \lambda\Sigma(t))\, dt + \sqrt{\Sigma(t)}\, dW(t)$$
$$d\Sigma(t) = (\Omega^2 + 2M\Sigma(t))dt + 2\sqrt{\Sigma(t)}(q_1\, dW(t) + q_2\, dZ(t))$$

where λ, M, Ω are now scalars. In this case, the instantaneous correlation between risky asset returns and the variance process Σ is $q_1/\sqrt{q_1^2+q_2^2}$. Parameter q_1 controls the sign of this correlation. In addition, the sum $q_1^2 + q_2^2$ determines both the level of this correlation and the volatility of volatility. The higher $|q_2|$, the lower the absolute correlation between return and volatility shocks, and the higher the volatility of volatility. □

To characterize the portfolio choice implications of process (4) it is convenient to use the infinitesimal generator $\mathcal{A}$ of the process Σ. Since the joint process $(\Sigma_{11}, \Sigma_{22}, \Sigma_{12})$ is just a trivariate diffusion process, $\mathcal{A}$ is defined in the standard way, as e.g. in Merton (1969), for functions $\phi = \phi(\Sigma)$. Using the particular structure of the dynamics (4) one can additionally show that $\mathcal{A}$ can be written in a very compact and simple matrix form. Precisely, let $\phi = \phi(\Sigma)$ be a smooth function. Then, the generator $\mathcal{A}$ associated with the diffusion process (4) takes the form:

$$\mathcal{A}\phi = tr\left\{\left(\Omega\Omega' + M\Sigma + \Sigma M'\right)\mathcal{D}\phi + 2\Sigma\mathcal{D}(Q'Q\mathcal{D}\phi)\right\} \tag{7}$$

where $tr(\cdot)$ is the trace operator and $\mathcal{D}$ is the matrix of differential operators.[8] In this form, it is obvious that this operator is affine in Σ, since the argument of the trace is affine in Σ; see also Bru (1991) and da Fonseca, Grasselli and Tebaldi (2005). This property of the generator $\mathcal{A}$ is important in order to obtain closed-form solutions of the Bellman equation describing the optimal portfolio problem with stochastic covariance matrix. We will show that this solution is affine in $\Sigma(t)$.

C. The Solution of the Investment Problem

The first challenge of solving the investment problem (3) subject to the covariance matrix dynamics (4), is that markets are incomplete because of the stochastic covariance matrix. Therefore, a multiplicity of equivalent martingale measures exists. We consider the value function implied by the min-max martingale measure. He and Pearson (1991) show that this value function can be characterized in terms of the following static problem:[9]

$$J(X_0, \Sigma_0) = \inf_{\nu} \sup_{\pi} \mathbb{E}\left[\frac{X(T)^\gamma - 1}{\gamma}\right] \tag{8}$$

$$\text{s.t.} \quad \mathbb{E}\left[\xi_\nu(T)X(T)\right] \leq x \tag{9}$$

where ν indexes the set of all equivalent martingale measures in our model and ξ_ν is in the set of associated state price densities. In what follows, we characterize the value function of problem (8)–(9). To obtain the result, we first show that process (4) is an affine process not only under the physical measure, but also under the min-max martingale measure. Moreover, it is possible to show that the terminal wealth is an exponentially affine function of Σ. We then exploit the fact that the indirect marginal utility of wealth is an exponentially affine function of Σ, with coefficients obtained as solutions of a system or matrix Riccati equations. These equations can be solved in closed form.[10] The final result is summarized in the next Proposition.

Proposition 2 *Given the covariance matrix dynamics (4), the value function of problem (3) takes the form*

$$J(X_0, \Sigma_0) = \frac{X_0^\gamma \widehat{J}(0, \Sigma_0)^{1-\gamma} - 1}{\gamma}$$

where the function $\widehat{J}(t, \Sigma_0)$ is given by[11]

$$\widehat{J}(t, \Sigma_0) = \exp\left(B(t,T) + \mathrm{tr}\left(A(t,T)\,\Sigma_0\right)\right) \tag{10}$$

with $B(t,T)$ and the symmetric matrix-valued function $A(t,T)$ solving the system of matrix Riccati differential equations:

$$0 = \frac{dB}{dt} + \mathrm{tr}[A\Omega\Omega'] - \frac{\gamma}{\gamma-1} r \tag{11}$$

$$0 = \frac{dA}{dt} + \Gamma' A + A\Gamma + 2A\Lambda A + C \tag{12}$$

under the terminal conditions $B(T,T) = 0$ and $A(T,T) = 0$, where

$$\Gamma = M - \frac{\gamma}{\gamma-1} Q' \begin{pmatrix} \lambda_1 & \lambda_2 \\ 0 & 0 \end{pmatrix}, \quad \Lambda = Q' \begin{pmatrix} 1 & 0 \\ 0 & 1-\gamma \end{pmatrix} Q, \quad C = \frac{\gamma}{2(\gamma-1)^2} \begin{pmatrix} \lambda_1^2 & \lambda_1\lambda_2 \\ \lambda_2\lambda_1 & \lambda_2^2 \end{pmatrix}$$

The solution of the system of matrix Riccati differential equations (11)-(12) is given in the proof (see Appendix B).

Remark. In the literature on affine term structure models, it is well known that modeling correlated stochastic factors is not straightforward. Duffie and Kan (1996) show that parametric restrictions on the drift matrix of the factor dynamics have to be satisfied for the existence of a regular affine process. In particular, out of diagonal elements in this matrix must have the same sign. This feature restricts the correlation structures that can be fitted with these models (see Duffee (2002)). In the Dai and Singleton (2000) classification for affine $A_m(n)$ models, specific restrictions need to be imposed for the model to be solvable: the Gaussian factors are allowed to be correlated, but the correlation between Gaussian and square-root factors need to be zero. This issue is well known also in the portfolio choice literature.[12] An interesting by-product of the results of Proposition 2 is that it provides a simple solution for the portfolio problem (3) without imposing additional restrictions on the dependence structure between the risk factors.

One advantage of the exponentially affine form of function $\widehat{J}$ in Proposition 2 is that it allows for a simple description of the partial derivatives of the marginal indirect utilities of wealth with respect to the variance and covariance factors. This property allows us to provide a very simple and easy to interpret solution to the incomplete-markets portfolio choice problem.

Proposition 3 *Let π be the optimal portfolio obtained under the assumptions of Proposition 2. It then follows,*

$$\pi = \frac{\lambda}{1-\gamma} + 2\begin{pmatrix} q_{11}A_{11} + q_{12}A_{12} \\ q_{12}A_{22} + q_{11}A_{12} \end{pmatrix} \tag{13}$$

where A_{ij} denotes the $ij-th$ component of matrix A which characterizes the indirect utility function $\widehat{J}(t, \Sigma_0)$.

The portfolio policy π is the sum of a myopic demand and a hedging demand. The hedging demands for variance and covariance risk are simple linear functions of the elements of the matrix A. The hedging demands on the first and the second asset are, respectively: $2(q_{11}A_{11} + q_{12}A_{12})$ and $2(q_{11}A_{12} + q_{12}A_{22})$. The interpretation is simple and it can be linked to the Merton's (1969) solution: the matrix A, describes how the component $\widehat{J}(t, \Sigma_0)$ of the indirect utility is affected by the state variables driving the dynamics of $\Sigma(t)$. Each element A_{ij} can be rewritten as $A_{ij} = \left[-X\frac{\partial J}{\partial X} / \frac{\partial^2 J}{\partial X^2}\right] \left(\frac{\partial J}{\partial X}\right)^{-1} \frac{\partial^2 J}{\partial \Sigma_{ij} \partial X}$, namely the product of the inverse of the relative risk aversion $-X\frac{\partial J}{\partial X} / \frac{\partial^2 J}{\partial X^2}$ multiplied by the sensitivity of the partial derivatives of the indirect utility $\frac{\partial J}{\partial X}$ with respect to the state variable $\Sigma(t)$, scaled by the current value of the marginal utility. Hedging demands are due to both a volatility hedging and a covariance hedging motive. Terms proportional to A_{11} and A_{22} are *pure* volatility hedging demands, deriving from the own volatility risk of assets one and two, respectively. Terms proportional to A_{12} are covariance hedging demands: they are due both to changes in volatility (when correlations are different from zero) and to changes in correlations. The second set of parameters that drive the optimal policy are q_{11}, q_{12}, and q_{22}. They determine the sign of the correlations between returns, variances and covariances.

Despite the simple structure of the hedging policies (13), a quite rich set of possible hedging demands can arise. For instance, if q_{11} and q_{12} are both negative, then volatility and correlation leverage effects will arise for all assets. If in addition A_{11}, A_{22} and A_{12} are all positive, then all hedging demands for variance and covariance risk will be negative and the exposure to all risky assets is reduced by the desire to hedge intertemporally variance and covariance risk. This is the situation we find in the calibration of our model to international equity returns. If, however, the sign of the marginal utility sensitivities, or the sign the correlations between returns, variances and covariances, is mixed, then it is possible to obtain some hedging demand components that are positive and some others that are negative. This is the setting we encounter, e.g., in the model calibration with Treasury Bill and corporate bond returns, where the optimal hedging demand for Treasury Bills is positive and the one for corporate bonds is negative.

In Proposition 2, the value function for the portfolio problem is written as a function of current wealth X and the components of the covariance matrix Σ. This parametrization enables us to isolate easily the hedging demand components for variance and covariance risk, but not the demand for correlation risk. The problem is that the hedging demand for covariance risk is caused by both volatility and correlation risk. Therefore, we can further split this demand in two further hedging components. A volatility hedging demand that hedges changes in returns covariance due to changes in assets' volatilities, and a correlation hedging demand. Such a decomposition enables us to quantify the contribution of correlation hedging to the overall hedging demand. Using Proposition 2, it is straightforward to compute these demands by using the fact that $\Sigma_{12} = \rho\sqrt{\Sigma_{11}\Sigma_{22}}$.

Proposition 4 *Let π be the optimal portfolio obtained under the assumptions of Proposition 2. The hedging demand for asset i is the sum of three components π_i^{vol}, $\pi_i^{vol/cov}$, π_i^{ρ}, which hedge, respectively, pure volatility risk, covariance risk due to volatilities, and correlation risk. The explicit expressions for these hedging demands are as follows.*

1. *Pure Volatility hedging:*
$$\pi_1^{vol} = 2q_{11}A_{11}, \qquad \pi_2^{vol} = 2q_{12}A_{22} \tag{14}$$

2. *Covariance hedging due to volatility:*
$$\pi_1^{vol/cov} = 2q_{11}A_{12}\rho\sqrt{\frac{\Sigma_{22}}{\Sigma_{11}}}, \qquad \pi_2^{vol/cov} = 2q_{12}A_{12}\rho\sqrt{\frac{\Sigma_{11}}{\Sigma_{22}}} \tag{15}$$

3. *Correlation hedging:*
$$\pi_1^{\rho} = 2A_{12}\left(q_{12} - q_{11}\rho\sqrt{\frac{\Sigma_{22}}{\Sigma_{11}}}\right), \qquad \pi_2^{\rho} = 2A_{12}\left(q_{11} - q_{12}\rho\sqrt{\frac{\Sigma_{11}}{\Sigma_{22}}}\right) \tag{16}$$

The pure volatility hedging demands in equation (14) for assets one and two are proportional to A_{11} and A_{22}, respectively. Their sign depends on the correlation between volatility shocks and returns, via the coefficients q_{11} and q_{12}, and the sign of the sensitivity of the marginal utilities of wealth to changes in volatility. For instance, when A_{11} and A_{22} are positive these sensitivities are positive, i.e. volatility is a bad state, and positive (negative) hedging demands for volatility arise if and only if returns and volatility shocks are positively (negatively) correlated. Covariance hedging demands due to volatility in equation (15) are proportional to the correlation level and the sensitivity of the marginal utilities of wealth to changes in covariances. This is intuitive, higher correlations imply a higher impact of a unit change in volatility on covariances, and imply a larger risk of an adverse covariance movement, deriving from a pure change in assets volatilities. In contrast to volatility hedging, the sign of the covariance hedging demand depends on the sign of the sensitivity of the marginal utility of wealth to changes in covariances, i.e. the sign of A_{12}. Depending on the given portfolio setting, this sign can easily be both positive or negative. For instance, in settings of low average correlations, in which portfolio diversification is the main objective, an increase in the correlation between assets typically implies a utility loss for the optimal portfolio, and a positive sensitivity of the marginal utility of wealth to covariances. This is, for instance, the situation we find in the calibration of our model to international equity returns. However, in a setting of large positive correlations, in which optimal portfolios typically imply spread positions between assets, the opposite might happen. In these settings, a higher correlation increases the gains of the spread position, implying a negative sensitivity of the marginal utility of wealth to changes in correlations. This is the situation we find, e.g., when we calibrate the model to Treasury Bills and corporate bond returns. The correlation hedging demand in equation (16) is proportional to the sensitivity of the marginal utility of wealth to changes in covariances. This is intuitive: the larger A_{12}, the larger the sensitivity of the marginal utility of wealth to correlation shocks, and the stronger the correlation hedging motive. Depending on the sign of q_{11} and q_{12}, correlation hedging can be increasing or decreasing in the correlation level and the ratio of the volatilities of the two assets. For example, π_1^ρ is increasing in ρ and $\sqrt{\Sigma_{22}/\Sigma_{11}}$ when q_{11} is negative, but is decreasing in the opposite case. The relative importance of π_i^ρ and $\pi_i^{vol/col}$ for the two assets depends on the relative size of q_{12} and $q_{11}\rho\sqrt{\Sigma_{11}/\Sigma_{22}}$, and q_{11} and $q_{12}\rho\sqrt{\Sigma_{11}/\Sigma_{22}}$, respectively. Even the signs of π_i^ρ and $\pi_i^{vol/col}$ can be the same or different, depending on the parameter configuration. In the next section, we make use of Proposition 4 to characterize the size, the sign and the extent of time variation of correlation hedging in realistic portfolio selection settings.

III. Hedging Correlation Risk

In order to quantify the size and the time variation of correlation hedging in realistic portfolio choice settings, we calibrate our model to real data. We start by studying two portfolio choice scenarios. The first scenario is the one of an international equity portfolio manager. For this calibration, we consider a portfolio of US and German stock indexes. In this case, the average correlation between equity markets is about 37% (see the top panel in Figure 2 for a description). We investigate how correlation risk affects the desire to diversify optimally international equity risk. The second scenario explores the case of a market-neutral hedge fund that uses spread trades to maximize return. The investor tries to build a near-arbitrage portfolio using two risky assets that are almost perfectly positively correlated. In this case, the investor tries to optimally hedge the risk of a leveraged position in one asset, using a corresponding short position in another asset. For this setting, we consider 10 years Treasury Bills and 10 years Aaa corporate bonds. Treasury Bills and Aaa corporate bonds yields had a very high average correlation of about 94% before 1998 and after 2003. The average correlation over the whole sample is 88% (see the bottom panel in Figure 2 for a description). In both cases, we obtain the time series of the conditional covariance matrix for our calibrations by estimating Engle's (2002) multivariate DCC model. Figures 2 and 3 show the estimated conditional correlations and volatilities.

Scenario I: International equity diversification. For equities, we use daily data for the S&P100 and the DAX indexes, from January 1988 to December 2005. Panel A of Table 1 presents the estimated unconditional moments of returns, volatilities and correlations for equities.

Insert Table 1 about here.

During this period, the unconditional mean of US and German stock returns is about 11% and 13%. The higher unconditional volatility of German returns arises together with a higher volatility of volatility. The unconditional correlation between the two stock indices is about 37% and the unconditional volatility of correlation is about 15%. This feature generates an obvious incentive for diversification.

Scenario II: Market neutral spread trading. For bonds, we use daily data of 10 years Treasury bills and Aaa corporate bond return indexes supplied by Lehman Brothers, from April 1996 to December 2005.[13] The average return on Aaa bonds in Panel B of Table 1 is about 1.84%, slightly higher than the 1.65% mean return of Treasury bills. The unconditional correlation is about 88% and is more than the double of the one between the S&P100 and the DAX index returns. The volatility of bonds correlation is about 9.5% and is approximately two third the volatility of stock returns correlation. The unconditional volatility and the volatility of volatility of Treasury bills are slightly higher than those of corporate bonds. These features generate an obvious incentive for exploiting near-arbitrage opportunities between the Treasury and the corporate bond markets. Therefore, the optimal portfolio in the second setting is very different, in nature, from the one in the first setting. In the first setting, investors try to exploit the low average correlation to diversify risk between the US and the German equity markets, by means of two long positions in the corresponding assets. In this case, correlation hedging will tend to hedge unanticipated changes in the correlation structure that might reduce the benefits of international diversification. In the second setting, investors will try to exploit the large average correlation and the low correlation volatility to develop a near arbitrage strategy between the Treasury and the corporate bond markets, by means of a long position in corporate bonds hedged by a short position in Treasuries. In this case, correlation hedging will tend to hedge unanticipated changes in correlations that might reduce the effectiveness of the near-arbitrage strategy.

In both examples, the average level and volatility of the correlation process are key parameters that affect the size of correlation hedging. Moreover, changes in returns correlations will cause the optimal correlation hedging demand to change over time. We study the size and the degree of time variation of correlation hedging in the following sections.

A. The Size of Correlation Hedging

We first calibrate the coefficient vector λ to match unconditional average risk premia, given estimates for the unconditional covariance matrix of returns. We then calibrate the covariance process (4) to the unconditional second moments of volatilities and correlations. Using calibrated parameters, we can then decompose the total hedging demand for asset $i = 1, 2$ in a pure volatility hedging component π_i^{vol}, a covariance hedging component for volatility $\pi_i^{vol/cov}$, and a correlation hedging demand π_i^{corr}, as described in Proposition 4. The calibrated parameters M and Q for the covariance matrix dynamics in equation (4) under the two portfolio scenarios are presented in Table 2.

Insert Table 2 about here.

The calibrated components q_{11} and q_{12} of the first row of the matrix Q are negative in the international equity diversification scenario. In this setting, the model implies a volatility leverage effect and a correlation leverage effect for both the returns of the US and the German equity markets. In the market neutral spread trading scenario, the calibrated parameter q_{11} is negative, but the calibrated parameter q_{12} is negative. Model implied correlation leverage effects arise only for the

Treasury Bills index returns. The relation between corporate bond index returns and correlations is, instead, positive. Similarly, volatility leverage effects arise for Treasury Bill and not for the corporate bond index returns. In particular, this parameter configuration implies that when correlations decrease, the spread between corporate bonds and Treasury Bill returns increase, consistent with the "flight to quality effect" from corporate bonds to the less risky Treasury Bill.

The negative coefficients in matrix M for stocks and bonds reflect the mean reversion in variances and covariances. The weaker mean reversions for stocks is due to the higher unconditional volatility of volatility and correlation.

To compute the average size of correlation hedging, covariance hedging due to volatility, and pure volatility hedging, as well as the comparative statics with respect to the average correlation and the correlation volatility, we initialize $\Sigma(t)$ at its unconditional value. We then compute the optimal hedging demand components π_i^{vol}, $\pi_i^{vol/cov}$, π_i^{corr}, $i = 1, 2$, as defined in Proposition 4, when the average correlation and the correlation volatility deviate from their sample value. We study the case in which the investment horizon is five years and the relative risk aversion parameter is $1 - \gamma = 3$. All calibrated hedging components are expressed in percentage of the corresponding absolute myopic Merton portfolio.

Scenario I: International equity diversification. In this first calibration, we move the average correlation over a grid in the interval $[0.25, 0.5]$, while holding fixed the remaining unconditional moments of returns. Consistently with the literature on univariate portfolio selection with stochastic volatility, pure volatility hedging is a small fraction of the myopic portfolio: on average its absolute size is never more than 3% of the myopic portfolio, as illustrated in Panels 2 and 4 (dotted lines) of Figure 8.

Insert Figure 8 about here.

The overall hedging demand for US and German stocks is dominated by correlation hedging, which is a significant fraction of the myopic portfolio: The hedging demand for correlation risk is about 10% of the myopic portfolio at the sample average correlation and it increases up to 13% for average correlations around 48% (see Panels 1 and 3). The absolute size of the correlation hedging is increasing in the average correlation. This is intuitive: When the average correlation increases, the available risky assets are less able to span the risk due to unexpected movements in the returns covariance matrix. Shocks causing a change in the conditional correlation can be hedged less effectively so that the absolute sensitivity of the marginal utility of wealth to the stochastic correlation is larger when the average correlation between returns is high. The covariance hedging component due to volatility in Panels 2 and 4 (straight lines) is on average about 4% of the myopic portfolio, and is more than the double of the pure volatility hedging component. This suggests that the implications of volatility hedging in a multivariate setting with stochastic correlations might be less negligible than in the univariate setting. In the international equity diversification scenario, the calibrated parameters imply a leverage effect on volatilities and correlations for all assets. In addition, all marginal utility sensitivities are positive. Therefore, all hedging demand components are negative and cumulate in the same direction. The cumulated hedging demand is on average 16% of the myopic demand at the sample average correlation, and about 20% of the myopic portfolio for average correlations around 48%.

In Figure 9, we present the comparative statics of the hedging demands with respect to the correlation volatility.

Insert Figure 9 about here.

We consider values for the unconditional correlation volatility over a grid in the interval $[0.05, 0.24]$, while holding fixed the other unconditional moments. The absolute size of correlation hedging is increasing in the correlation volatility: for correlation volatilities of about 24%, correlation hedging in US and German stocks can be as large as 14% of the myopic portfolio (see Panels 1 and 3). For correlation volatilities near zero, correlation hedging tends to vanish. This is intuitive: A higher correlation volatility implies a higher risk that the myopic portfolio will be ex-post sub-optimal. The pattern of the pure volatility hedging demand in Panels 2 and 4 (dotted lines) is flatter that the one of the hedging demand for correlation risk. Moreover, for both US and German stocks, the volatility hedging component is no larger than 3.5% of the myopic portfolio. This finding further confirms in our multivariate framework the small size of pure volatility hedging demands found in univariate dynamic portfolio choice settings (see, e.g., Chako and Viceira, 2005).

Scenario II: Market neutral spread trading. To obtain comparative statics results for market neutral spread trades, we consider values for the average correlation of bond returns over a grid in the interval $[0.75, 0.98]$. Pure volatility hedging is a small fraction of the myopic portfolio: its absolute size in Panels 6 and 8 of Figure 8 (dotted lines) is never larger than approximately 3% of the myopic portfolio. The hedging components against covariance risk caused by volatility changes in the same panels (straight lines) is on average 6% of the myopic portfolio. The overall hedging demand in bonds is dominated by correlation hedging, as illustrated by Panels 5 and 7 of Figure 8: correlation hedging is about 14% of Merton's myopic portfolio for unconditional correlations around 0.87 and about 16% for a correlation of 0.98. As expected, correlation hedging is increasing in the average correlation level. The intuition is that when the average correlation is close to one, the available risky assets are less able to span the risk due to unexpected changes in the returns covariance matrix. Shocks reducing the conditional correlation, thus increasing the risk of the market neutral spread trade, can be hedged less effectively when the average correlation between returns is high. Therefore, the marginal utility of terminal wealth is more sensitive to changes in the correlation level. For the calibrated parameters, the model implies a correlation leverage effect for Treasury Bills, but a positive sign for the relation between corporate bond returns and correlations. At the same time, we obtain a volatility leverage effect on corporate bond returns and an opposite relation between volatility and Treasury Bill returns. A further distinguishing feature from the international equity diversification scenario is that the marginal utility sensitivities to volatility are positive for both asset classes, but the sensitivity to correlations and covariances is negative. Therefore, the market neutral spread trading scenario implies a positive (negative) correlation hedging demand for Treasury Bills (corporate bonds). In a similar vein, the hedging demand for covariance risk due to volatility are positive (negative) for Treasury bills (corporate bonds). Overall, the total positive/negative hedging demand for Treasury Bill/corporate bonds is on average 20% of the myopic portfolio. The aggregate hedging effect on the total portfolios reduces the spread position between Treasury Bills and corporate bonds implied by the optimal myopic demand.

To explore the effect of the correlation volatility, we consider values in the interval $[0.01, 0.16]$, while holding fixed the remaining moments. Volatility hedging is a quite flat function of the correlation volatility, both for Treasuries and corporate bonds, and is on average 2% of the myopic portfolio (see Figure 9, Panels 6 and 8, dotted lines). The hedging component for covariance risk due to volatility is larger and is about 6% of the myopic demand (Panels 6 and 8, straight lines). The absolute size of the correlation hedging demand can be as large as 16% of the myopic portfolio, for Treasuries and corporate bonds, when the correlation volatility is about 16% (see Panels 5 and 7). As expected, we also find that it is an increasing function of the correlation volatility, which tends to vanish at zero, value at which correlation risk also vanishes. The intuition is clear. A higher correlation volatility implies a higher risk in exploiting existing near-arbitrage opportunities, because the long-short strategy can be ex-post risky. In this case case, the wealth marginal utility sensitivity to changes in the correlation level is larger.

A.1. Time horizon

An important question addressed by the optimal portfolio choice literature is how the optimal allocation in risky assets varies with respect to the investment horizon. This question is of key importance for investment professionals working in the pension fund industry and for individuals deciding the composition of their retirement accounts. Brennan, Schwartz, and Lagnado (1997), Barberis (2000), Kim and Omberg (1996), and Wachter (2002) address this issue in the context of time-varying expected returns. When volatilities are constant, they find that the optimal investment in risky assets increases in the investment horizon. Kim and Omberg (1996) show that for the investor with utility over terminal wealth and $\gamma > 1$, when volatilities are constant the optimal allocation increases in the investment horizon as long as the risk premium is positive. Wachter (2002) extends this result to the case of utility over intertemporal consumption. This result hinges on the assumption of no uncertainty about the correlation structure of asset returns. Depending on the characteristics of this uncertainty, however, it is reasonable to expect that the optimal demand for hedging correlation risk could mitigate, if not offset, the speculative components. Our model offers a simple theoretical framework to investigate the role of uncertainty on the nature of this relationship. In the international equity diversification scenario, we find that the total optimal allocation to risky assets decreases with the investment horizon. In what follows we quantify this effect.

The correlation hedging component is increasing in the investment horizon. At a six month horizon, the correlation hedging component is on average about -1% of the myopic international diversified portfolio. This component increases to -10% at a five year horizon and -23% at a eight year horizon. The longer the horizon, the larger is the correction. It is interesting to compare the optimal demand to hedge correlation risk with the demand to hedge pure volatility risk. For instance, in the case of German index returns and at a six month horizon the two components are both very small and similar (-0.3% and -0.15%, respectively). At a 8 year horizon, the difference is quite striking (-22% and -3%, respectively). The results are summarized in Table 3 and show that in some cases the results change quite substantially when correlation risk is taken into account.

Insert Table 3 about here.

B. The Time Variation of Correlation Hedging

Using the calibrated parameters in Section $A.$, we can study the time variation of correlation hedging, as a function of changes in the covariance matrix $\Sigma(t)$. For simplicity, we make use of the in-sample time series of conditional covariance matrices estimated from the DCC model to depict a realistic trajectory of the covariance matrix process. Given such a trajectory, we use the formulas in Proposition 4 to obtain the estimated time-variation in the different hedging demand components. Since pure volatility hedging is constant in our setting, we focus on the hedging portfolio for correlation risk and the hedging portfolio for covariance risk due to changes in volatility.

Scenario I. International equity diversification: Figure 10 plots over time the optimal correlation hedging demand, and the hedging portfolio for covariance risk due to changes in volatility, in the stock example.

Insert Figure 10 about here.

Panels 1 and 3 present the correlation hedging policies for the US and German stock indexes. Both policies display significant time variation, especially during periods of higher correlation variability, and follow quite closely the realized correlation dynamics. Correlation hedging portfolios for US and German stocks tend to vary quite smoothly over time, following the smooth time variation of correlations in Panel 5. Over time, correlation hedging in US stocks attains peaks of about 12% at the begin of the sample, whereas correlation hedging in German stocks reaches a maximum of about 11% at the same time. Compared to these numbers, the hedging portfolio for covariance risk due to changes in volatility is much smaller.

Scenario II. Market neutral spread trading: Hedging policies in the bonds example are presented in Figure 11 and further support the major role of correlation hedging.

Insert Figure 11 about here.

The hedging portfolio for covariance risk due to changes in volatility in Panel 2 and 4 is most of the time less than about 6.5% of the myopic portfolio. Correlation hedging in Treasury and corporate bonds is larger. It closely follows the time variation of bond correlations in Panel 3. The largest time variations are observed in periods of market distress, after the Russian crisis of 1998. Starting from 1998 and until after the end of the nineties, optimal correlation hedging often implies a sizable reduction of the hedging portfolio in Treasury and corporate bonds, which closely follows the reduced conditional correlation. The average correlation hedging demand in Treasuries and corporate bonds is about 14% of the myopic demand. It is often more than this level for Treasuries and corporate bonds, but can also shrink to lower levels of about 11% in the case of corporate bonds, when correlations collapse to levels of less than about 40% and 50%, as for instance in the years 1998, 2000 and 2003. During these large "flight to quality" events, when correlation collapses, correlation hedging in Panel 1 and 3 decreases. Over time, correlation hedging generates an increasing excess demand for Treasuries ex-ante, which has the goal of hedging a possible "flight to quality" event, i.e. a fall in the correlation between Treasury Bills and corporate bond returns, ex-post.

IV. Extensions and Robustness

A. Utility over consumption

The previous results focus on the case of an agent maximizing utility of terminal wealth. It is interesting to explore how these results change in the case of an investor maximizing utility over intertemporal consumption. When markets are complete and volatility is constant, Wachter (2002) shows analytically that the optimal portfolio weights of a CRRA investor maximizing intertemporal consumption are equal to a weighted average of the portfolio weights solving a sequence of individual terminal wealth problems. In this context, wealth can be interpreted as a bond that pays consumption as its coupon and the value of wealth is equal to the sum of the "zeros" that pay the optimal consumption at each date. For instance, if an agent is concerned not just about its retirement account but also about the purchase of a house, the optimal portfolio is a weighted average of two separate portfolios: a "retirement portfolio" and a "house portfolio". When markets are incomplete, the previous analytical result in general does not hold. However, under the min-max martingale measure, markets are completed by the addition of a set of securities that span the state-space receiving zero portfolio weights in equilibrium (He and Pearson (1991)). Under this specific martingale measure, the main results of the previous section still hold and the portfolio demand for hedging correlation risk would still be an important component of the optimal portfolio. Clearly, since the duration of lifetime consumption is lower than the investment horizon of the agent maximizing utility over terminal wealth, the absolute demand of risky assets would be proportionally lower.

B. More general dynamic settings

The previous sections studied correlation risk for a portfolio choice problem, in which expected excess returns are linearly related to the covariance of returns (Cox, Ingersoll, and Ross (1985), Heston (1993)), interest rates are constant, and the covariance process, although stochastic, is autonomous. In what follows, we investigate the implications of these three assumptions with regards to the properties of the optimal portfolio policies. We consider an economy in which: First, the risk premium is constant. Second, interest rates are stochastic and are a function of the same state variables driving asset prices. Third, additional risk factors, like the interest rate or an aggregate market indicator, can affect the dynamics properties of the covariance matrix $\Sigma(t)$ (i.e. its speed of mean reversion and volatility).[14] The economy that we consider implies that the square Sharpe ratios are a decreasing function of the returns covariance matrix, and therefore it allows a direct comparison with the results in Chacko and Viceira (2005), who solve in closed-form the equivalent univariate problem for the case with a constant interest rate. Under these assumptions, it is known (see Chacko and Viceira (2005)) that to obtain closed-form solutions it is convenient to model directly the information matrix (i.e. Σ^{-1}), as opposed to Σ. Let Σ^{-1} be affine in a $3\times3-$dimensional Wishart process Y:

$$\Sigma^{-1} = SYS'$$

where the 2×3 matrix S is such that $SS' = id_{2\times2}$.[15] The process Y satisfies the Wishart dynamics:

$$dY(t) = [\Omega\Omega' + MY(t) + Y(t)M']dt + Y^{1/2}(t)dBQ + Q'dB'Y^{1/2}(t) \tag{17}$$

where the matrices Ω, M and Q are now of dimension 3×3 and where $B = [W, Z_1, Z_2]$, with Z_1 and Z_2 being each a three-dimensional Brownian motion, independent of the first three-dimensional Brownian motion W. It then follows that $\Sigma = SY^{-1}S'$, and it is natural to define $\Sigma^{1/2}$ as the 2×3 matrix $SY^{-1/2}$.[16] In this way, we obtain a return process given by

$$dS(t) = I_S \left[\begin{pmatrix} r(t) + \mu_1^e \\ r(t) + \mu_2^e \end{pmatrix} dt + \Sigma^{1/2}(t)dW(t) \right] \tag{18}$$

where the excess return vector $\mu^e = (\mu_1^e, \mu_2^e)' \in \mathbb{R}^2$ is constant and the stochastic risk-less rate $r(t)$ is defined by the affine functional form:

$$r(t) = r_0 + tr(RY(t)) \tag{19}$$

where $r_0 > 0$ and R is a 3×3 matrix. Notice that the non-negativity of $r(t)$ is ensured simply by assuming that the symmetric matrix R is positive definite. This setting is effectively a six-factor model, in which some interest rate risk factors might be linked to the covariance matrix of stock returns, in dependence of the specific form of the matrix D in equation (**??**). It is straightforward to verify that the square Sharpe ratio in this model is affine in Y. Therefore, the portfolio choice problem (3) can be solved in closed form in this extended dynamic setting as well.

Proposition 5 *The solution of the portfolio problem (3) for the returns dynamics (17)-(18) and under a stochastic interest rate (19) is:*

$$J(X_0, Y_0) = \frac{X_0^\gamma \widehat{J}(0, Y_0)^{1-\gamma} - 1}{\gamma}$$

where

$$\widehat{J}(0, Y_0) = \exp\left(B(t,T) + \mathrm{tr}\left(A(t,T)Y_0\right)\right)$$

with $B(t,T)$ and the symmetric matrix-valued function $A(t,T) := A_1(t,T) + A_2(t,T)$ solving the following system of matrix Riccati differential equations, which can be solved in closed form:

$$-\frac{dB}{dt} = -\frac{\gamma}{\gamma-1}r_0 + tr(A\Omega\Omega') \tag{20}$$

$$-\frac{dA_i}{dt} = \Gamma_i' A_i + A_i'\Gamma_i + 2A_i'\Lambda_i A_i + C_i \tag{21}$$

where $i = 1, 2$. In these equations, the coefficients Γ_1, Γ_2 are given by:

$$\Gamma_1 = MS'S - \frac{\gamma}{\gamma-1}Q'e_1\mu^{e\prime}S \quad , \quad \Gamma_2 = M(id_{3\times3} - S'S)$$

The coefficients Λ_1, Λ_2 are given by

$$\Lambda_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1-\gamma & 0 \\ 0 & 0 & 1-\gamma \end{pmatrix} \quad , \quad \Lambda_2 = \begin{pmatrix} 1-\gamma & 0 & 0 \\ 0 & 1-\gamma & 0 \\ 0 & 0 & 1-\gamma \end{pmatrix}$$

and the coefficients C_1, C_2 are:

$$C_1 = \frac{\gamma}{2(\gamma-1)^2}S'\mu^e\mu^{e\prime}S - \frac{\gamma}{\gamma-1}S'SDS'S \quad , \quad C_2 = -\frac{\gamma}{\gamma-1}(id_{3\times3} - S'S)D(id_{3\times3} - S'S)$$

Finally, the optimal policy for this portfolio problem reads:

$$\pi = \frac{1}{1-\gamma}\Sigma^{-1}\begin{pmatrix} \mu_1^e \\ \mu_2^e \end{pmatrix} + 2\Sigma^{-1}S\begin{pmatrix} q_{11}A_{11} + q_{12}A_{12} + q_{13}A_{13} \\ q_{11}A_{21} + q_{12}A_{22} + q_{13}A_{23} \\ q_{11}A_{31} + q_{12}A_{32} + q_{13}A_{33} \end{pmatrix} \tag{22}$$

The optimal policy (22) consists of a myopic and an intertemporal hedging portfolio, which are both proportional to the stochastic inverse covariance matrix. As noted by Chacko and Viceira (2005), in the univariate setting the relative size of the hedging and myopic demands is independent of the current level of volatility. This property holds also in the multivariate case. However, the hedging demand depends also on the correlations between asset returns. We calibrate the Wishart process Y to the specification used to model the covariance matrix Σ, for the special case $R = 0$ (constant interest rate) and under the restriction of an autonomous covariance matrix dynamics, and compare the results with the findings in the previous sections. We find that the average relative size of hedging and myopic demands in Proposition 5 are similar to the ones implied by the setting in Proposition 2. At least for the calibrated economy, the results are robust to this alternative specification of the expected returns. The intuition for this result follows from the fact that the key economic variable, which determines the optimal hedging portfolio, is the squared conditional Sharpe ratio of asset returns. From the proof of Proposition 2 and Proposition 5, it is easy to see that the sensitivity of the marginal utility of wealth to the underlying state variables is the expectation of the exponential of an integrated square Sharpe ratio. In the setting of Proposition 2, this quantity is defined as $\lambda'\Sigma\lambda = \lambda'\Sigma\Sigma^{-1}\Sigma\lambda$. In the setting of Proposition 5 the square Sharpe ratio is $\mu^e\Sigma^{-1}\mu^e$. Therefore, if λ is such that $\mu^e = \mathbb{E}(\Sigma(t))\lambda$, where $\mathbb{E}(\Sigma(t))$ is the unconditional first moment of $\Sigma(t)$, the relative size of the hedging and myopic portfolios in the two settings should be expected to be similar.

V. Correlation risk, Inflation and Pension Funds

There exists a broad variety of portfolio choice settings, in which one naturally expects correlations between returns to be stochastic, and correlation hedging to be quantitatively and qualitatively important. A natural application of our model arises in settings where inflation-linked instruments like index-linked bonds are explicitly incorporated in the optimal portfolio, to hedge – to some extent – the inflation risk of a nominal investment. Examples of such portfolio problems include the problem of a pension fund trying to optimize the risk-adjusted real surplus of the invested portfolio, in excess of the final real value of some fixed contribution plan. Even if index-linked bonds offer some protection against inflation risk, the correlation of their returns with the inflation level is far from perfect. In the U.S., this correlation ranges from about 0.13 at a monthly frequency to about 0.31 at a yearly frequency. In the UK, where the correlation level is the highest, these correlations are 0.17 at a monthly frequency and about 0.68 at a yearly frequency. Moreover, the correlation between the returns of index-linked bonds and other non-indexed instruments is time-varying. Figure 13 plots the estimated conditional volatilities of (top panel) and the correlations between (bottom panel) 10 years Treasury Bills and 10 year TIPS daily index returns between April 1998 and December 2005.

Insert Figure 13 about here.

The average correlation in this time period is about 0.7. The time variation of the correlation is, however, large and it ranges between 0.3 at the end of the nineties and 0.85 at the begin of 2004. Moreover, as for the previous portfolio choice examples, correlation changes are quite persistent, and are more persistent than changes in returns volatilities. This has immediate implications for a pension fund manager who uses these two instruments to manage inflation risk.

Consider for example a U.S. pension fund manager investing in a portfolio consisting of cash, 10-year maturity Treasury Bills and 10-year maturity indexed bonds (TIPS). We calibrate our model to the times series of Treasury Bills and TIPS daily returns, volatilities and correlations. Figure 12 shows the model-implied hedging demands.

Insert Figure 12 about here.

The average size of the correlation hedging demand in the two top left panels of Figure 12 is about 12.5%. Similar to the previous portfolio scenario with Treasury Bills and corporate bonds, correlation hedging tends to reduce the myopic spread position between TIPS and Treasury bonds, by increasing the demand for Treasuries and reducing the demand for TIPS. The optimal myopic spread position between TIPS and Treasuries is due to the high Sharpe ratio of index-linked bonds, which is mainly due to their very low returns volatility. The average hedging demand for covariance risk due to volatility, in the two top right panels of Figure 12, is about 5.5%. These demands have the same sign as the corresponding correlation hedging demands. Therefore, the sum of the two hedging components of the single assets is on average about 18% of the corresponding myopic portfolios. We can compute the empirical certainty equivalent loss of a suboptimal investment strategy in cash, TIPS and Treasuries, from the perspective of an investor starting investing in April 1998, and having December 2005 as the final date of her investment horizon. The certainty equivalent loss of a myopically optimal strategy, relative to the optimal dynamic one, is about 8.18% of the initially invested wealth. These findings show that correlation is an important risk factor and it should be explicitly considered in a broad range of optimal portfolio choice, including those typically addressed by pension fund, hedge fund, and asset-liability managers.

VI. Discussion and Conclusions

Apart from the two main scenarios that we have studied in this article, correlation risk plays an important role in a large variety of financial settings. An example is the pricing, hedging, and risk management of quantos. A Quanto is a type of financial instrument in which the underlying is denominated in one currency, but the instrument itself is settled in another currency at some fixed rate. Such products are attractive for portfolio managers or hedge funds who wish to have exposure to a foreign asset, but without carrying the corresponding exchange rate risk. Well known examples include differential swaps (also known as quantity-adjusted swaps, guaranteed exchange rate swaps, Libor differential swaps), quanto options, quanto equity swaps, and quanto futures (such as the Nikkei Future traded on the CME). Essentially, a quanto has an embedded currency forward with a variable notional amount. It is this variable notional amount that gives quantos their name—"quanto" is short for "quantity adjusting option". In these cases, the pricing, hedging, and risk management of these instruments depend directly on the correlation between the risk factors (See Reiner (1992) and Dravid, Richardson and Sun (1993)). For these reasons, these instruments are also referred by practitioners as "correlation products". In differential swaps, for instance, the dealer commits to pay a floating rate on a fixed US \$ notional principal amount, rather than on a fixed amount in the foreign currency, as with a typical cross-currency swap. This commitment exposes the dealer to changes in the correlation between the Libor and the exchange rates. Since static hedging strategies are generally not feasible, the dealer must manage the residual correlation risk by using optimal portfolio techniques modeling correlation risk.

Correlation risk play also an important role in the trading of multi-asset options. A multi-asset option is a derivative security whose payoff depends on the future values of several possibly correlated underlying assets. Typical examples are basket, spread, outside barrier options and options on the minimum/maximum of several assets. Since equity correlation risk cannot be hedged as precisely as volatility risk, the current market practice of equity derivatives desks is simply to monitor their correlation exposure and try to avoid risk peaks in certain correlation pairs, by managing the product flow via dynamic price margins, and by using index options to hedge the 'average' correlation risk within the index basket. However, specific correlation risk in individual stock pairs remains. The results of this article provide an alternative strategy to manage the correlation risk.

Correlation risk is also important in the credit derivative market as the likelihood of a default of one credit may affect the likelihood of default of another (default correlation). Examples of correlation-based products include also instruments written on baskets of credits, such as CDOs and first-to-default (FTD) swaps. Investors in FTD structures sell protection on a reference portfolio of names and assume exposure to the first default to take place within the pre-defined basket of credits. On occurrence of a default, the protection seller needs to indemnify the protection buyer for the loss on the defaulted credit calculated on the basis of the full face value of the FTD basket. Clearly, the ex-ante value of these instruments is very sensitive to the correlations structure in the basket.

To summarize, in this article we have analyzed the importance of correlation risk for portfolio decisions and proposed a new simple approach to embed stochastic correlation in optimal portfolio choice. The flexibility of the model allows us to study correlation hedging in settings that can account for several stylized features of asset returns, including volatility/correlation leverage effects and volatility clustering phenomena. We show that correlation hedging contributes to the overall portfolio in a qualitatively and quantitatively significant way. Examples of calibrated economies show that correlation hedging can be as large as about 16% of Merton's myopic portfolio, whereas volatility hedging is typically very small. Correlation hedging demand is highly time varying and is larger for settings with extreme average correlations and higher correlation variances.

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Notes

[1]Since the available instruments for investment consist of two assets, the resulting portfolio optimization problem is effectively univariate, because the budget constraint allows to eliminate one portfolio weight in the optimization problem.

[2]For n assets, one needs to model $\frac{n(n+1)}{2}$ element of the covariance matrix, which implies that the matrix A and B have $\frac{1}{4}n^2(n+1)^2$ elements.

[3]The explicit expression for the correlation dynamics is derived in Appendix A.

[4]It is assumed negative semi-definite, in order to ensure the typical mean reverting behaviour and stationarity.

[5]Details on the calibration are provided in Section **III.**.

[6]Gourieroux and Sufana (2003) apply a setting with Wishart volatilities and no leverage effects to credit derivatives pricing.

[7]Figure 7 has been obtained with the same parameterizations used to obtain Figure 6.

[8]The matrix differential operator $\mathcal{D}$ is given by $\mathcal{D} := \begin{pmatrix} \frac{\partial}{\partial \Sigma_{11}} & \frac{\partial}{\partial \Sigma_{12}} \\ \frac{\partial}{\partial \Sigma_{21}} & \frac{\partial}{\partial \Sigma_{22}} \end{pmatrix}$.

[9]See also Pliska (1986) and Cox and Huang (1988) for the Markovian complete markets case.

[10]See, e.g., Reid (1972) for a review of Riccati differential equations.

[11]Give a square matrix M, $tr(M)$ is the trace of M.

[12]Liu (2005) addresses this issue by assuming a triangular factor structure in an affine portfolio problem with two risky assets.

[13]The same exercise applied to a sample of monthly bond returns over a longer sample starting in January 1988 yielded similar results.

[14]Ang and Bekaert (2002).

[15]This parametrization admits all 2×3 selection matrices, like e.g.:

$$S = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

In this case, $SS' = id_{2\times 2}$ and SYS' is the 2×2 upper diagonal sub-block of Y.

[16]A proof of this statement is presented in Appendix A.

Appendix A: Proofs of Theorems

Proof of Proposition 1: The dynamics of the correlation process implied by the Wishart covariance matrix diffusion (4) is computed using Itô's Lemma. Let

$$\rho(t) = \frac{\Sigma_{12}(t)}{\sqrt{\Sigma_{11}(t)\Sigma_{22}(t)}} \tag{A1}$$

be the instantaneous correlation between the returns of the first and the second risky assets and denote by σ_{ij} and q_{ij} the $ij-$th component of the volatility matrix $\Sigma^{1/2}$ and the matrix Q in equation (4), respectively. Applying Itô's Lemma to (A1) and using the dynamics for Σ_{11}, Σ_{22} and Σ_{12}, implied by (4), it follows:

$$\begin{aligned}
d\rho = & \left[-\frac{\rho}{2}\left(\frac{q_{11}^2+q_{21}^2}{\Sigma_{11}}+\frac{q_{12}^2+q_{22}^2}{\Sigma_{22}}\right)+(\rho^2-2)\frac{q_{11}q_{12}+q_{21}q_{22}}{\sqrt{\Sigma_{11}\Sigma_{22}}}+(1-\rho^2)\frac{m_{21}\Sigma_{11}+m_{12}\Sigma_{22}}{\sqrt{\Sigma_{11}\Sigma_{22}}}\right]dt \\
& -\left[\frac{\rho}{\Sigma_{11}\Sigma_{22}}\left(\Sigma_{22}\sigma_{11}q_{11}+\Sigma_{11}\sigma_{12}q_{12}\right)-\frac{\sigma_{12}q_{11}+\sigma_{11}q_{12}}{\sqrt{\Sigma_{11}\Sigma_{22}}}\right]dW_1 \\
& -\left[\frac{\rho}{\Sigma_{11}\Sigma_{22}}\left(\Sigma_{11}\sigma_{22}q_{12}+\Sigma_{22}\sigma_{21}q_{11}\right)-\frac{\sigma_{22}q_{11}+\sigma_{21}q_{12}}{\sqrt{\Sigma_{11}\Sigma_{22}}}\right]dW_2 \\
& -\left[\frac{\rho}{\Sigma_{11}\Sigma_{22}}\left(\Sigma_{22}\sigma_{11}q_{21}+\Sigma_{11}\sigma_{12}q_{22}\right)-\frac{\sigma_{11}q_{22}+\sigma_{12}q_{21}}{\sqrt{\Sigma_{11}\Sigma_{22}}}\right]dZ_1 \\
& -\left[\frac{\rho}{\Sigma_{11}\Sigma_{22}}\left(\Sigma_{11}\sigma_{22}q_{22}+\Sigma_{22}\sigma_{21}q_{21}\right)-\frac{\sigma_{21}q_{22}+\sigma_{22}q_{21}}{\sqrt{\Sigma_{11}\Sigma_{22}}}\right]dZ_2
\end{aligned} \tag{A2}$$

Therefore, the instantaneous drift of the correlation process is a quadratic polynomial with state dependent coefficients:

$$\mathbb{E}\left[d\rho(t)\middle|\mathcal{F}_t\right] = A(t)\,\rho(t)^2 + B(t)\,\rho(t) + C(t) \tag{A3}$$

where coefficients $A(t)$, $B(t)$ and $C(t)$ are given by:

$$A(t) = \frac{q_{11}q_{12}+q_{21}q_{22}}{\sqrt{\Sigma_{11}(t)\Sigma_{22}(t)}} - m_{21}\sqrt{\frac{\Sigma_{11}(t)}{\Sigma_{22}(t)}} - m_{12}\sqrt{\frac{\Sigma_{22}(t)}{\Sigma_{11}(t)}} \tag{A4}$$

$$B(t) = -\frac{1}{2}\left(\frac{q_{11}^2+q_{21}^2}{\Sigma_{11}(t)}+\frac{q_{12}^2+q_{22}^2}{\Sigma_{22}(t)}\right) \tag{A5}$$

$$C(t) = -2\frac{q_{11}q_{12}+q_{21}q_{22}}{\sqrt{\Sigma_{11}(t)\Sigma_{22}(t)}} + m_{21}\sqrt{\frac{\Sigma_{11}(t)}{\Sigma_{22}(t)}} + m_{12}\sqrt{\frac{\Sigma_{22}(t)}{\Sigma_{11}(t)}} \tag{A6}$$

The instantaneous conditional variance of the correlation process is easily obtained from equation (A2) using the independence of Brownian motions Z and W, and it is a third order polynomial with state dependent coefficients:

$$\mathbb{E}\left[d\rho(t)^2\middle|\mathcal{F}_t\right] = \left(1-\rho^2(t)\right)\left(\frac{q_{11}^2+q_{21}^2}{\Sigma_{11}(t)}+\frac{q_{12}^2+q_{22}^2}{\Sigma_{22}(t)}+2\rho(t)\frac{q_{11}q_{12}+q_{21}q_{22}}{\sqrt{\Sigma_{11}(t)\Sigma_{22}(t)}}\right)$$

This concludes the proof.

Proof of Proposition 2: Since the covariance matrix dynamics (4) is stochastic and depends on the bivariate Brownian motion Z, markets are incomplete. In incomplete markets, several martingale measures exist, each of them associated with a particular market price of risk generated by Z. Let

$$\Theta_\nu = \Sigma^{1/2} [\ \lambda \quad \nu\ ] = \Sigma^{1/2}(\lambda e_1' + \nu e_2'), \tag{A7}$$

be the matrix of market prices of risk associated with the matrix valued Brownian motion $B = [W, Z]$, where $\nu \in \mathbb{R}^2$, $e_1 = (1,0)'$ and $e_2 = (0,1)'$. Given Θ_ν, the associated martingale measure implies a process ξ_ν of stochastic discount factors, defined for $t \in [0,T]$ by:

$$\xi_\nu(t) = e^{-rt - tr\left(\int_0^t \Theta_\nu'(s)\, dB(s) + \frac{1}{2}\int_0^t \Theta_\nu'(s)\Theta_\nu(s) ds\right)} \tag{A8}$$

In general, the choice of ν is arbitrary and different choices imply different optimal portfolios. We choose the min-max martingale measure to solve the portfolio problem. This choice is due to two reasons. First, this measure has the economic property that if the market were completed by the addition of two assets, the optimal investment in these securities would be zero. Second, the min-max measure is the only measure that admits an equivalent static representation of the initial dynamic portfolio problem by means of the following dual problem (He and Pearson, 1991):

$$J(x, \Sigma_0) = \inf_\nu \sup_\pi \mathbb{E}\left[\frac{X(T)^\gamma - 1}{\gamma}\right] \tag{A9}$$

$$\text{s.t.} \quad \mathbb{E}\left[\xi_\nu(T) X(T)\right] \leq x \tag{A10}$$

where $X(0) = x$. In what follows, we focus on the solution of problem (A9)-(A10). The optimality conditions for the innermost maximization is:

$$X(T) = (\psi \xi_\nu(T))^{\frac{1}{\gamma-1}} \tag{A11}$$

where the Lagrange multiplier for the static budget constraint is

$$\psi = x^{\gamma-1}\mathbb{E}\left[\xi_\nu(T)^{\frac{\gamma}{\gamma-1}}\right]^{1-\gamma}.$$

It then follows:

$$J(x, \Sigma_0) = x^\gamma \inf_\nu \frac{1}{\gamma}\mathbb{E}\left[\xi_\nu(T)^{\frac{\gamma}{\gamma-1}}\right]^{1-\gamma} - \frac{1}{\gamma} \tag{A12}$$

Using (A8) and (A12), one can notice that the solution requires the computation of the expected value of the exponential of a stochastic integral. A simple change of measure reduces the problem to the calculation of the expectation of the exponential of a deterministic integral. Let P^γ be the probability measure defined by the following Radon-Nykodim derivative with respect to the physical measure P:

$$\frac{dP^\gamma}{dP} = e^{-tr\left(\frac{\gamma}{\gamma-1}\int_0^T \Theta_\nu'(s)\, dB(s) + \frac{1}{2}\frac{\gamma^2}{(\gamma-1)^2}\int_0^T \Theta_\nu'(s)\Theta_\nu(s) ds\right)} \tag{A13}$$

We denote expectations under P^γ by $\mathbb{E}^\gamma[\cdot]$. Then, the minimizer of (A12) is the solution of the following problem:[17]

$$\begin{aligned}
\widehat{J}(x, \Sigma_0) &= \inf_\nu \mathbb{E}\left[\xi_\nu(T)^{\frac{\gamma}{\gamma-1}}\right] \\
&= \inf_\nu \mathbb{E}^\gamma\left[e^{-\frac{\gamma}{\gamma-1}rT + \frac{\gamma}{2(\gamma-1)^2} tr\left(\int_0^T \Theta_\nu'(s)\Theta_\nu(s) ds\right)}\right] \\
&= \inf_\nu \mathbb{E}^\gamma\left[e^{-\frac{\gamma}{\gamma-1}rT + \frac{\gamma}{2(\gamma-1)^2} tr\left(\int_0^T \Sigma(s) ds (\lambda\lambda' + \nu\nu')\right)}\right]
\end{aligned} \tag{A14}$$

Notice that the expression in the exponential of the expectation in (A14) is affine in Σ. By Girsanov Theorem, under the P^γ measure the stochastic process B^γ, defined as

$$B^\gamma(t) = B(t) + \frac{\gamma}{\gamma-1}\int_0^t \Theta_\nu(s) ds$$

is a 2×2 matrix of standard Brownian motions. Therefore, the process (4) is an affine process also under new probability measure (P^γ, B^γ):

$$\begin{aligned}
d\Sigma(t) = &\left[\Omega\Omega' + \left(M - \frac{\gamma}{\gamma-1}Q'(e_1\lambda' + e_2\nu')\right)\Sigma(t) + \Sigma(t)\left(M - \frac{\gamma}{\gamma-1}Q'(e_1\lambda' + e_2\nu')\right)'\right] dt \\
&+ \Sigma^{1/2}(t) dB^\gamma(t) Q + Q' dB(t)^{\gamma\prime}\Sigma^{1/2}(t)
\end{aligned} \tag{A15}$$

Using Feynman Kac formula, it is known that if the optimal ν and $\widehat{J}$ solve the probabilistic problem (A14), then they must also be a solution of the following Hamilton Jacobi Bellman (HJB) equation:

$$0 = \frac{\partial \widehat{J}}{\partial t} + \inf_\nu \left\{\mathcal{A}\widehat{J} + \widehat{J}\left[-\frac{\gamma}{\gamma-1}r + \frac{\gamma}{2(\gamma-1)^2}tr(\Sigma(\lambda\lambda' + \nu\nu')\right]\right\} \quad , \tag{A16}$$

subject to the terminal condition $\widehat{J}(\Sigma,T)=1$, where $\mathcal{A}$ is the infinitesimal generator of the matrix-valued diffusion (A15), which is given by:

$$\begin{aligned}\mathcal{A} &= tr\left(\left(\Omega\Omega' + \left(M - \frac{\gamma}{\gamma-1}Q'(e_1\lambda' + e_2\nu')\right)\Sigma + \Sigma\left(M - \frac{\gamma}{\gamma-1}Q'(e_1\lambda' + e_2\nu')\right)'\right)\mathcal{D}\right)\\ &\quad + tr(2\Sigma\mathcal{D}Q'Q\mathcal{D}) \end{aligned} \tag{A17}$$

where

$$\mathcal{D} := \begin{pmatrix} \frac{\partial}{\partial\Sigma_{11}} & \frac{\partial}{\partial\Sigma_{12}} \\ \frac{\partial}{\partial\Sigma_{21}} & \frac{\partial}{\partial\Sigma_{22}} \end{pmatrix} \tag{A18}$$

The generator is affine in Σ. The optimality condition for the optimal control ν, implied by HJB equation (A16), is:

$$\frac{1}{\gamma-1}\Sigma\nu = \frac{\partial}{\partial\nu}tr\left(\left(Q'(\nu e_2')'\Sigma + \Sigma\nu e_2' Q\right)\frac{\mathcal{D}\widehat{J}}{\widehat{J}}\right) = \frac{\partial}{\partial\nu}tr\left(\frac{\mathcal{D}\widehat{J}}{\widehat{J}}Q'(\nu e_2')'\Sigma + \Sigma\nu e_2' Q\frac{\mathcal{D}\widehat{J}}{\widehat{J}}\right)$$

Applying rules for the derivative of trace operators, the right hand side can be written as $\Sigma\left(\frac{\mathcal{D}\widehat{J}}{\widehat{J}} + \frac{\mathcal{D}\widehat{J}'}{\widehat{J}}\right)Q'e_2$. It follows that

$$\nu = (\gamma-1)\left(\frac{\mathcal{D}\widehat{J}}{\widehat{J}} + \frac{\mathcal{D}\widehat{J}'}{\widehat{J}}\right)Q'e_2$$

We now compute the generator (A17) associated with this solution. To this end, note that

$$\lambda e_1' + \nu e_2' = \lambda e_1' + (\gamma-1)\left(\frac{\mathcal{D}\widehat{J}}{\widehat{J}} + \frac{\mathcal{D}\widehat{J}'}{\widehat{J}}\right)Q'e_2e_2'$$

Substituting the expression for ν obtained in equation (A17), we obtain the generator

$$\begin{aligned}\mathcal{A} &= tr\left(\left(\Omega\Omega' + \left(M - \frac{\gamma}{\gamma-1}Q'e_1\lambda'\right)\Sigma + \Sigma\left(M - \frac{\gamma}{\gamma-1}Q'e_1\lambda'\right)'\right)\mathcal{D} + 2\Sigma\mathcal{D}Q'Q\mathcal{D}\right)\\ &\quad - \gamma\, tr\left(\left(Q'e_2e_2'Q\left(\frac{\mathcal{D}\widehat{J}}{\widehat{J}} + \frac{\mathcal{D}\widehat{J}'}{\widehat{J}}\right)\Sigma + \Sigma\left(\frac{\mathcal{D}\widehat{J}}{\widehat{J}} + \frac{\mathcal{D}\widehat{J}'}{\widehat{J}}\right)Q'e_2e_2'Q\right)\mathcal{D}\right)\\ &= tr\left(\left(\Omega\Omega' + \left(M - \frac{\gamma}{\gamma-1}Q'e_1\lambda'\right)\Sigma + \Sigma\left(M - \frac{\gamma}{\gamma-1}Q'e_1\lambda'\right)'\right)\mathcal{D} + 2\Sigma\mathcal{D}Q'Q\mathcal{D}\right)\\ &\quad - \gamma\widehat{J}\, tr\left(\Sigma\left(\frac{\mathcal{D}\widehat{J}}{\widehat{J}} + \frac{\mathcal{D}\widehat{J}'}{\widehat{J}}\right)Q'e_2e_2'Q\left(\frac{\mathcal{D}}{\widehat{J}} + \frac{\mathcal{D}'}{\widehat{J}}\right)'\right)\end{aligned}$$

Substitution of the last expression for $\mathcal{A}$ into the HJB equation (A16) yields the following partial differential equation for $\widehat{J}$:

$$\begin{aligned}-\frac{\partial\widehat{J}}{\partial t} &= tr\left(\left(\Omega\Omega' + \left(M - \frac{\gamma}{\gamma-1}Q'e_1\lambda'\right)\Sigma + \Sigma\left(M - \frac{\gamma}{\gamma-1}Q'e_1\lambda'\right)'\right)\mathcal{D} + 2\Sigma\mathcal{D}Q'Q\mathcal{D}\right)\widehat{J}\\ &\quad + \frac{\gamma}{\gamma-1}\widehat{J}\left(-r + \frac{tr\,(\Sigma\lambda\lambda')}{2(\gamma-1)}\right) - \frac{\gamma}{2}\widehat{J}\, tr\left(\Sigma\left(\frac{\mathcal{D}\widehat{J}}{\widehat{J}} + \frac{\mathcal{D}\widehat{J}'}{\widehat{J}}\right)Q'e_2e_2'Q\left(\frac{\mathcal{D}\widehat{J}}{\widehat{J}} + \frac{\mathcal{D}\widehat{J}'}{\widehat{J}}\right)'\right)\end{aligned}$$

subject to the boundary condition $\widehat{J}(\Sigma,T)=1$. Clearly, the affine structure of this problem suggests an exponentially affine functional form for its solution:

$$\widehat{J}(t,\Sigma) = \exp(B(t,T) + tr(A(t,T)\Sigma))$$

for some state independent coefficients $B(t,T)$ and $A(t,T)$. After inserting this functional form into the differential equation for $\widehat{J}$, the guess can be easily verified. The coefficients B and A are the solutions of the following system of Riccati equations:

$$\begin{aligned}-\frac{dB}{dt} &= tr(A\Omega\Omega') - \frac{\gamma}{\gamma-1}r\\ -tr\left(\frac{dA}{dt}\Sigma\right) &= tr\left(\Gamma'A\Sigma + A\Gamma\Sigma + 2AQ'QA\Sigma - \frac{\gamma}{2}(A'+A)Q'e_2e_2'Q(A'+A)\Sigma + \frac{\gamma}{2(\gamma-1)^2}\lambda\lambda'\Sigma\right)\end{aligned}$$

with terminal conditions $B(T,T)=0$ and $A(T,T)=0$, where

$$\Gamma = M - \frac{\gamma}{\gamma-1} Q' e_1 \lambda'$$

For a symmetric matrix function A, the second differential equation implies the following matrix Riccati equation:

$$0 = \frac{dA}{dt} + \Gamma' A + A\Gamma + 2AQ'(id_{2\times 2} - \gamma e_2 e_2')QA + \frac{\gamma}{2(\gamma-1)^2}\lambda\lambda' \tag{A19}$$

Defining

$$\Lambda = Q'(id_{2\times 2} - \gamma e_2 e_2')Q = Q' \begin{pmatrix} 1 & 0 \\ 0 & 1-\gamma \end{pmatrix} Q \quad , \quad C = \frac{\gamma}{2(\gamma-1)^2}\lambda\lambda' \tag{A20}$$

the system of matrix Riccati equations in the statement of Proposition 2 is obtained. These differential equations are completely integrable, so that closed-form expressions for $\widehat{J}$ (and hence for J) can be computed.

Given a solution for A, function B is obtained by simple integration:

$$B(t,T) = \frac{\gamma}{\gamma-1} r\,(T-t) - tr\left(\int_t^T A(s,T)\Omega\,\Omega'\,ds\right)$$

The solution for A is:

$$A(t,T) = \widetilde{A} + e^{\Theta'(T-t)}\left[-\widetilde{A}^{-1} + 2\int_t^T e^{\Theta(T-s)}\Lambda e^{\Theta'(T-s)} ds\right]^{-1} e^{\Theta(T-t)} \tag{A21}$$

where $\Theta = \Gamma + 2\Lambda\tilde{A}$, $e^{\cdot}$ is the matrix exponential function and matrix $\widetilde{A}$ is the solution of the quadratic system

$$\Gamma'\,\widetilde{A} + \widetilde{A}\,\Gamma + 2\widetilde{A}\Lambda\widetilde{A} + C = 0 \tag{A22}$$

To prove that (A22) is indeed the solution for A, we apply the variation of constants method. Let $\tau = T - t$ denote time to maturity. With this variable, the Riccati differential equation for A is:

$$\frac{dA}{d\tau} = \Gamma' A(\tau) + A(\tau)\Gamma + 2A(\tau)\Lambda A(\tau) + C\ , \tag{A23}$$

subject to initial condition $A(0)=0$. Let A^* be the solution of the following quadratic matrix equation

$$\Gamma' A^* + A^*\Gamma + 2A^*\Lambda A^* + C = 0$$

and define $A(\tau) := K(\tau) + A^*$. After inserting this guess for A into equation (A23), we see that $K(\tau)$ must solve the matrix differential equation

$$\frac{dK}{d\tau} = \Gamma^{*'} K(\tau) + K(\tau)\Gamma^* + 2K(\tau)\Lambda K(\tau), \tag{A24}$$

subject to initial condition $K(0) = -A^*$, where $\Gamma^* := \Gamma + 2\Lambda A^*$. Define

$$K(\tau) = \exp(\Gamma^*\tau)' H(\tau)\exp(\Gamma^*\tau)$$

for some matrix valued function $H(\tau)$. It then follows,

$$\begin{aligned}\frac{dK}{d\tau} &= \Gamma^{*'}\exp(\Gamma^*\tau)'H(\tau)\exp(\Gamma^*\tau) + \exp(\Gamma^*\tau)'H(\tau)\exp(\Gamma^*\tau)\Gamma^* + \exp(\Gamma^*\tau)'\frac{dH(\tau)}{d\tau}\exp(\Gamma^*\tau) \\ &= \Gamma^{*'} K(\tau) + K(\tau)\Gamma^* + \exp(\Gamma^*\tau)'\frac{dH(\tau)}{d\tau}\exp(\Gamma^*\tau)\end{aligned}$$

By comparing the last expression with (A24), the function $H(\tau)$ must satisfy the following matrix differential equation:

$$\frac{dH}{d\tau} = 2H(\tau)\exp(\Gamma^*\tau)\Lambda\exp(\Gamma^*\tau)'H(\tau)\ ,$$

subject to initial condition $H(0) = -A^*$. After integrating this equation, using the definition of $K(\tau)$ and recalling that $A(\tau) = K(\tau) + A^*$, we obtain (A22). This concludes the proof.

Proof of Proposition 3: In order to recover the optimal portfolio policy we have, from the proof of Proposition 2:

$$X^*(t) =: \frac{1}{\xi_{\nu^*}(t)}\mathbb{E}[\xi_{\nu^*}(T)X^*(T)\,|\mathcal{F}_t] = \psi^{\frac{1}{\gamma-1}}\xi_{\nu^*}(t)^{\frac{1}{\gamma-1}}\,\widehat{J}(t,\Sigma(t)) \tag{A25}$$

For the Wishart dynamics (4), Itô's lemma applied to both sides of (A25) gives, for every state Σ:

$$X^*(t)\,tr\left(\begin{bmatrix}\pi_1 & \pi_2\\ 0 & 0\end{bmatrix}\Sigma^{1/2}dB\right) = X^*(t)\,tr\left(\frac{1}{1-\gamma}\Theta_{\nu^*}'dB + \frac{\mathcal{D}\widehat{J}'}{\widehat{J}}\left(\Sigma^{1/2}dBQ + Q'dB'\Sigma^{1/2}\right)\right) \tag{A26}$$

This implies

$$\Sigma^{1/2}\begin{bmatrix}\pi_1 & 0\\ \pi_2 & 0\end{bmatrix} = \frac{1}{1-\gamma}\,\Theta_{\nu^*} + 2\Sigma^{1/2}AQ'$$

We conclude that portfolio weight $\pi = (\pi_1, \pi_2)'$ is

$$\pi = \frac{\lambda}{1-\gamma} + 2AQ'e_1 = \frac{1}{1-\gamma}\begin{pmatrix}\lambda_1\\ \lambda_2\end{pmatrix} + 2\begin{pmatrix}q_{11}A_{11} + q_{12}A_{12}\\ q_{12}A_{22} + q_{11}A_{12}\end{pmatrix} \tag{A27}$$

This concludes the proof of the proposition.

Proof of Proposition 4: To obtain the optimal hedging demand in terms of the state variables Σ_{11}, Σ_{22}, and ρ, write $\Sigma_{12} = \rho\sqrt{\Sigma_{11}\Sigma_{12}}$ and note that:

$$\frac{-X}{\frac{\partial^2 J}{\partial^2 X}} \frac{\partial^2 J}{\partial \rho \partial X} = \frac{-X}{\frac{\partial^2 J}{\partial^2 X}} \frac{\partial^2 J}{\partial \Sigma_{12} \partial X} \frac{\partial \Sigma_{12}}{\partial \rho} = A_{12}\sqrt{\Sigma_{11}\Sigma_{22}} \tag{A28}$$

This is the closed form expression for the wealth-scaled ratio of marginal utilities with respect to ρ and X. For volatilities, the same argument gives, for $i, j = 1, 2$, where $i \neq j$:

$$\frac{-X}{\frac{\partial^2 J}{\partial^2 X}} \frac{\partial^2 J}{\partial \Sigma_{ii} \partial X} = A_{ii} + \frac{-X}{\frac{\partial^2 J}{\partial^2 X}} \frac{\partial^2 J}{\partial \Sigma_{12} \partial X} \frac{\partial \Sigma_{12}}{\partial \Sigma_{ii}} = A_{ii} + A_{12}\frac{\rho}{2}\sqrt{\frac{\Sigma_{jj}}{\Sigma_{ii}}} \tag{A29}$$

This is the closed form expression for the wealth-scaled ratio of marginal utilities with respect to Σ_{ii} and X, when ρ is treated as an explicit state variable, in addition to Σ_{11} and Σ_{22}. The first term on the right hand side of equation (A29) is the one that corresponds to the direct effect of Σ_{ii} on the value function. The second term is the one that correspond to the indirect effect of Σ_{ii}, via the feedback of Σ_{ii} on Σ_{12}. To compute the corresponding hedging demand it is then enough to use Merton (1969)'s results and to calculate the projection coefficients of $d\Sigma_{11}$, $d\Sigma_{22}$ and $d\rho$ on the space spanned by dS_1/S_1 and dS_2/S_2, using the available dynamics. After collecting terms proportional to $A_{12}\sqrt{\Sigma_{11}\Sigma_{22}}$, A_{11}, A_{22}, $A_{12}\frac{\rho}{2}\sqrt{\Sigma_{11}/\Sigma_{22}}$ and $A_{12}\frac{\rho}{2}\sqrt{\Sigma_{22}/\Sigma_{11}}$, respectively, the desired decomposition follows. This concludes the proof of the proposition.

Proof of Proposition 5: We first prove a useful technical result on the form of the inverse covariance matrix $\Sigma^{-1} = (SYS')^{-1}$ when $SS' = id_{2\times 2}$.

Lemma 1 *Let $SS' = id_{2\times 2}$. It then follows:*

$$(SYS')^{-1} = SY^{-1}S'$$

Proof of Lemma 1: Since SYS' is symmetric, we have:

$$SYS' = Q\Lambda Q' \quad , \quad (SYS')^{-1} = Q\Lambda^{-1}Q'$$

where Q is a 2×2 matrix of eigenvectors of SYS' and Λ a diagonal 2×2 matrix of eigenvalues. Similarly,

$$Y = \bar{Q}\bar{\Lambda}\bar{Q}' \quad , \quad Y^{-1} = \bar{Q}\bar{\Lambda}^{-1}\bar{Q}'$$

where $\bar{Q}$ is a 3×3 matrix of eigenvectors of Y and $\bar{\Lambda}$ a diagonal matrix of eigenvalues. We first show that the eigenvectors of SYS' are all vectors q_i such that $S'q_i$ is an eigenvector of Y. Indeed, let $\bar{q}_i = S'q_i$ be an eigenvector of Y. It then follows,

$$SYS'q_i = SY\bar{q}_i = \lambda_i S\bar{q}_i = \lambda_i q_i$$

where λ_i is an eigenvalue of both SYS and Y. In particular, the non-zero elements of Λ are a subset of the nonzero elements of $\bar{\Lambda}$. We also have, for all eigenvectors q_i of SYS:

$$S\bar{q}_i = SS'q_i = q_i$$

Since S has rank 2, one eigenvector $\bar{q}_i$ of SYS' must be such that $S\bar{q}_i = 0$. Without loss of generality, let this eigenvector be $\bar{q}_3$. We then have:

$$S\bar{Q} = \begin{bmatrix} Q & 0_{2\times 1} \end{bmatrix}$$

and

$$SY^{-1}S' = S\bar{Q}\bar{\Lambda}^{-1}\bar{Q}'S' = \begin{bmatrix} Q & 0_{2\times 1} \end{bmatrix}\bar{\Lambda}^{-1}\begin{pmatrix} Q' \\ 0_{1\times 2} \end{pmatrix} = Q\Lambda^{-1}Q'$$

because the non zero elements in Λ are a subset of those in $\bar{\Lambda}$. From (**??**), we conclude:

$$(SYS')^{-1} = SY^{-1}S'$$

as desired. This concludes the proof of the Lemma.

We now proceed with the proof of Proposition 5. Let $\nu = [\nu_1, \nu_2, \nu_3]$, where $\nu_1, \nu_2, \nu_3 \in \mathbb{R}^3$. It turns out, that the value function can be written in the form:

$$J(x, Y_0) = x^\gamma \inf_\nu \frac{1}{\gamma}\mathbb{E}\left[\xi_\nu(T)^{\frac{\gamma}{\gamma-1}}\right]^{1-\gamma} - \frac{1}{\gamma} = \frac{x^\gamma \widehat{J}(0, Y_0)^{1-\gamma} - 1}{\gamma}$$

subject to the constraint $S\nu_1 = 0$, where

$$\begin{aligned} \mathbb{E}\left[\xi_\nu(T)^{\frac{\gamma}{\gamma-1}}\right] &= \mathbb{E}^\gamma\left[e^{-\frac{\gamma}{\gamma-1}\int_0^T r(s)ds + \frac{\gamma}{2(\gamma-1)^2}tr(\int_0^T \Sigma(s)^{-1}ds\mu^e\mu^{e\prime} + \int_0^T Y(s)ds(\nu_1\nu_1' + \nu_2\nu_2' + \nu_3\nu_3'))}\right] \\ &= \mathbb{E}^\gamma\left[e^{-\frac{\gamma}{\gamma-1}(r_0 + tr(\int_0^T Y(s)dsD)) + \frac{\gamma}{2(\gamma-1)^2}tr(\int_0^T Y(s)ds(S'\mu^e\mu^{e\prime}S + \nu_1\nu_1' + \nu_2\nu_2' + \nu_3\nu_3'))}\right] \end{aligned} \quad \text{(A30)}$$

for a probability measure P^γ defined by the density:

$$\frac{dP^\gamma}{dP} = e^{-tr\left(\frac{\gamma}{\gamma-1}\int_0^T \Theta_\nu'(s)dB + \frac{1}{2}\frac{\gamma^2}{(\gamma-1)^2}\int_0^T \Theta_\nu'(s)\Theta_\nu(s)ds\right)}$$

The 3×3 matrix Θ_ν of market prices of risk is defined by:

$$\Theta_\nu = \begin{bmatrix} \Sigma^{1/2\prime}\Sigma^{-1}\mu^e & 0_{3\times 1} & 0_{3\times 1} \end{bmatrix} + \begin{bmatrix} \tilde{\nu}_1 & \tilde{\nu}_2 & \tilde{\nu}_3 \end{bmatrix}$$

where $\tilde{\nu}_1, \tilde{\nu}_2, \tilde{\nu}_3 \in \mathbb{R}^3$ and $\Sigma^{1/2}\tilde{\nu}_1 = 0$. For convenience, we parameterize Θ_ν as:

$$\Theta_\nu = Y^{-1/2}\begin{bmatrix} S'SYS'\mu^e & 0_{3\times 1} & 0_{3\times 1} \end{bmatrix} + Y^{1/2}\begin{bmatrix} \nu_1 & \nu_2 & \nu_3 \end{bmatrix}$$

where $\nu_i = Y^{-1/2}\tilde{\nu}_i$ for $i = 1, 2, 3$. In this parametrization, the constraint $\Sigma^{1/2}\tilde{\nu}_1 = 0$ reads $S\nu_1 = 0$, and the objective function (??)-(A30) follows. Similarly to the previous portfolio choice settings, the following decomposition of Θ_ν applies:

$$\Theta_\nu = Y^{-1/2}S'SYS'\mu^e e_1' + Y^{1/2}\nu$$

where e_i it the $i-$th unit vector in $\mathbb{R}^3$. The dynamics of Y under the probability P^γ are:

$$\begin{aligned} dY &= \Big[\Omega\Omega' + \left(M - \frac{\gamma}{1-\gamma}Q'\nu'\right)Y - \frac{\gamma}{1-\gamma}Q'e_1\mu^{e\prime}SYS'S \\ &+ Y\left(M - \frac{\gamma}{1-\gamma}Q'\nu'\right)' - \frac{\gamma}{1-\gamma}S'SYS'\mu^e e_1'Q\Big]dt + Y^{1/2}dBQ + Q'dB'Y^{1/2} \end{aligned}$$

These dynamics are affine in Y. It follows, that the infinitesimal generator of the process Y is:

$$\begin{aligned} \mathcal{A} &= tr\left(\left(\Omega\Omega' + \left(M - \frac{\gamma}{1-\gamma}Q'\nu'\right)Y - \frac{\gamma}{1-\gamma}Q'e_1\mu^{e\prime}SYS'S\right)\mathcal{D}\right) \\ &+ tr\left(\left(Y\left(M - \frac{\gamma}{1-\gamma}Q'\nu'\right)' - \frac{\gamma}{1-\gamma}S'SYS'\mu^e e_1'Q\right)\mathcal{D}\right) + tr\left(2Y\mathcal{D}Q'Q\mathcal{D}\right) \end{aligned} \tag{A31}$$

The HJB equation for $\widehat{J}$ can be then written as:

$$0 = \frac{\partial\widehat{J}}{\partial t} - \frac{\gamma}{\gamma-1}(r_0 + tr(YD)) + \inf_\nu\left\{\mathcal{A}\widehat{J} + \frac{\gamma}{2(\gamma-1)^2}tr\left(Y(S'\mu^e\mu^{e\prime}S + \nu_1\nu_1' + \nu_2\nu_2' + \nu_3\nu_3')\right)\right\}$$

subject to $S\nu_1 = 0$ and the terminal condition $\widehat{J}(T, Y) = 1$. For $i = 2, 3$, the optimal control ν_i is given by:

$$\nu_i = (\gamma - 1)\left(\frac{\mathcal{D}\widehat{J}}{\widehat{J}} + \frac{\mathcal{D}\widehat{J}'}{\widehat{J}}\right)Q'e_i$$

The optimal control ν_1 satisfies the optimality condition

$$\frac{1}{\gamma-1}Y\nu_1 = Y\left(\frac{\mathcal{D}\widehat{J}}{\widehat{J}} + \frac{\mathcal{D}\widehat{J}'}{\widehat{J}}\right)Q'e_1 + S'\lambda$$

where $\lambda \in \mathbb{R}^2$ is the vector of Lagrange multipliers of the constraint $S\nu_1 = 0$. This gives:

$$\nu_1 = (\gamma - 1)\left(\left(\frac{\mathcal{D}\widehat{J}}{\widehat{J}} + \frac{\mathcal{D}\widehat{J}'}{\widehat{J}}\right)Q'e_1 + Y^{-1}S'\lambda\right)$$

Using the constraint

$$0 = S\nu_1 = (\gamma - 1)\left(S\left(\frac{\mathcal{D}\widehat{J}}{\widehat{J}} + \frac{\mathcal{D}\widehat{J}'}{\widehat{J}}\right)Q'e_1 + SY^{-1}S'\lambda\right)$$

it follows:

$$\lambda = -(\gamma - 1)SYS'S\left(\frac{\mathcal{D}\widehat{J}}{\widehat{J}} + \frac{\mathcal{D}\widehat{J}}{\widehat{J}}\right)Q'e_1$$

Therefore, the optimal control ν_1 reads explicitly:

$$\nu_1 = (\gamma - 1)\left(id_{3\times 3} - Y^{-1}S'SYS'S\right)\left(\frac{\mathcal{D}\widehat{J}}{\widehat{J}} + \frac{\mathcal{D}\widehat{J}'}{\widehat{J}}\right)Q'e_1$$

It then follows:

$$\begin{aligned} \nu &= \nu_1 e_1' + \nu_2 e_2' + \nu_2 e_3' \\ &= (\gamma - 1)\left[\left(\frac{\mathcal{D}\widehat{J}}{\widehat{J}} + \frac{\mathcal{D}\widehat{J}'}{\widehat{J}}\right)Q' - Y^{-1}S'SYS'S\left(\frac{\mathcal{D}\widehat{J}}{\widehat{J}} + \frac{\mathcal{D}\widehat{J}'}{\widehat{J}}\right)Q'e_1e_1'\right] \end{aligned}$$

After inserting the expression for the optimal ν in equation (A31), we obtain the generator:

$$\begin{aligned} \mathcal{A} &= tr\left(\left(\Omega\Omega' + MY - \frac{\gamma}{1-\gamma}Q'e_1\mu^{e\prime}SYS'S + YM' - \frac{\gamma}{1-\gamma}S'SYS'\mu^e e_1'Q\right)\mathcal{D} + 2Y\mathcal{D}Q'Q\mathcal{D}\right) \\ &- \gamma tr\left(\left(Q'Q\left(\frac{\mathcal{D}\widehat{J}}{\widehat{J}} + \frac{\mathcal{D}\widehat{J}'}{\widehat{J}}\right)Y - Q'e_1e_1'Q\left(\frac{\mathcal{D}\widehat{J}}{\widehat{J}} + \frac{\mathcal{D}\widehat{J}'}{\widehat{J}}\right)S'SYS'S\right)\mathcal{D}\right) \\ &- \gamma tr\left(\left(Y\left(\frac{\mathcal{D}\widehat{J}}{\widehat{J}} + \frac{\mathcal{D}\widehat{J}'}{\widehat{J}}\right)Q'Q - S'SYS'S\left(\frac{\mathcal{D}\widehat{J}}{\widehat{J}} + \frac{\mathcal{D}\widehat{J}'}{\widehat{J}}\right)Q'e_1e_1'Q\right)\mathcal{D}\right) \end{aligned}$$

Using the properties of the trace operator, we obtain the more compact expression:

$$\begin{aligned}\mathcal{A} &= tr\left(\left(\Omega\Omega' + MY - \frac{\gamma}{1-\gamma}Q'e_1\mu^{e\prime}SYS'S + YM' - \frac{\gamma}{1-\gamma}S'SYS'\mu^e e_1'Q\right)\mathcal{D} + 2Y\mathcal{D}Q'Q\mathcal{D}\right)\\ &-\gamma\widehat{J}tr\left(\left(Y\left(\frac{\mathcal{D}\widehat{J}}{\widehat{J}} + \frac{\mathcal{D}\widehat{J}'}{\widehat{J}}\right)Q'Q - S'SYS'S\left(\frac{\mathcal{D}\widehat{J}}{\widehat{J}} + \frac{\mathcal{D}\widehat{J}'}{\widehat{J}}\right)Q'e_1e_1'Q\right)\left(\frac{\mathcal{D}}{\widehat{J}} + \frac{\mathcal{D}'}{\widehat{J}}\right)\right)\end{aligned}$$

Substitution of the last expression for $\mathcal{A}$ in the HJB equation (??), using the form of the optimal control ν and the identity $SS' = id_{2\times 2}$, yields the following partial differential equation for $\widehat{J}$:

$$\begin{aligned}-\frac{\partial\widehat{J}}{\partial t} &= tr\left(\left(\Omega\Omega' + MY - \frac{\gamma}{1-\gamma}Q'e_1\mu^{e\prime}SYS'S + YM' - \frac{\gamma}{1-\gamma}S'SYS'\mu^e e_1'Q\right)\mathcal{D} + 2Y\mathcal{D}Q'Q\mathcal{D}\right)\widehat{J}\\ &-\frac{\gamma}{2}\widehat{J}tr\left(\left(Y\left(\frac{\mathcal{D}\widehat{J}}{\widehat{J}} + \frac{\mathcal{D}\widehat{J}'}{\widehat{J}}\right)Q'Q - S'SYS'S\left(\frac{\mathcal{D}\widehat{J}}{\widehat{J}} + \frac{\mathcal{D}\widehat{J}'}{\widehat{J}}\right)Q'e_1e_1'Q\right)\left(\frac{\mathcal{D}\widehat{J}}{\widehat{J}} + \frac{\mathcal{D}\widehat{J}'}{\widehat{J}}\right)\right)\\ &-\frac{\gamma}{\gamma-1}(r_0 + tr(YD)) + \frac{\gamma}{2(\gamma-1)^2}tr(YS'\mu^e\mu^{e\prime}S)\end{aligned}$$

subject to the terminal condition $\widehat{J}(T,Y) = 1$. The affine structure of this problem suggests an affine functional form for $\widehat{J}$:

$$\widehat{J}(t,Y) = \exp\left(B(t,T) + tr(A(t,T)Y)\right)$$

where the functions $B(t,T)$ and $A(t,T)$ are state-independent and satisfy the boundary condition $B(T,T) = 0$ and $A(T,T) = 0$. It follows that B and A satisfy the following system of differential equations:

$$\begin{aligned}-\frac{dB}{dt} &= -\frac{\gamma}{\gamma-1}r_0 + tr(A\Omega\Omega')\\ -tr\left(\frac{dA}{dt}Y\right) &= tr\left(\left(M'A + AM - \frac{\gamma}{\gamma-1}(S'SAQ'e_1\mu^{e\prime}S + S'\mu^e e_1'QAS'S)\right)Y\right)\\ &+tr\left(\left(2((1-\gamma)AQ'QA + \gamma S'SAQ'e_1e_1'QAS'S) + \frac{\gamma}{2(\gamma-1)^2}S'\mu^e\mu^{e\prime}S - \frac{\gamma}{\gamma-1}D\right)Y\right)\end{aligned}$$

with terminal condition $B(T,T) = 0$ and $A(T,T) = 0$. In order to solve the equation for A, it is useful to apply the following decomposition:

$$\begin{aligned}\frac{dA}{dt}Y &= \frac{dA}{dt}S'SY + \frac{dA}{dt}(id_{3\times 3} - S'S)Y\\ &= \frac{dA}{dt}S'S(S'SY) + \frac{dA}{dt}(id_{3\times 3} - S'S)(id_{3\times 3} - S'S)Y\end{aligned}$$

We can use this decomposition to obtain two systems of matrix Riccati differential equations for $A_1 := AS'S$ and $A_2 := A(id_{3\times 3} - S'S)$. The system of equations for A_1 is:

$$\begin{aligned}-\frac{dA_1}{dt} &= S'SM'A_1 - \frac{\gamma}{\gamma-1}S'\mu^e e_1'QA_1 + A_1'MS'S - \frac{\gamma}{\gamma-1}A_1'Q'e_1\mu^{e\prime}S\\ &+2(1-\gamma)A_1'Q'QA_1 + \gamma A_1'Q'e_1e_1'QA_1 + \frac{\gamma}{2(\gamma-1)^2}S'\mu^e\mu^{e\prime}S - \frac{\gamma}{\gamma-1}S'SDS'S\\ &= \left(S'SM' - \frac{\gamma}{\gamma-1}S'\mu^e e_1'Q\right)A_1 + A_1'\left(MS'S - \frac{\gamma}{\gamma-1}Q'e_1\mu^{e\prime}S\right)\\ &+2A_1'Q'\left((1-\gamma)id_{3\times 3} + \gamma e_1e_1'\right)QA_1 + \frac{\gamma}{2(\gamma-1)^2}S'\mu^e\mu^{e\prime}S - \frac{\gamma}{\gamma-1}S'SDS'S\end{aligned}$$

subject to the terminal condition $A_1(T,T) = 0$. The system of matrix Riccati differential equations for A_2 is:

$$\begin{aligned}-\frac{dA_2}{dt} &= (id_{3\times 3} - S'S)M'A_2 + A_2'M(id_{3\times 3} - S'S) + 2(1-\gamma)A_2'Q'QA_2\\ &-\frac{\gamma}{\gamma-1}(id_{3\times 3} - S'S)D(id_{3\times 3} - S'S)\end{aligned}$$

subject to the terminal condition $A_2(T,T) = 0$. The unique solution for A is obtained as $A = A_1 + A_2$.

In order to recover the optimal portfolio policy, we follow similar arguments as in the proof of Proposition 3 and use the relation:

$$X^*(t) =: \frac{1}{\xi_{\nu^*}(t)}\mathbb{E}[\xi_{\nu^*}(T)X^*(T)\,|\mathcal{F}_t] = \psi^{\frac{1}{\gamma-1}}\xi_{\nu^*}(t)^{\frac{1}{\gamma-1}}\,\widehat{J}(t,Y(t)) \tag{A32}$$

Itô's lemma applied to both sides of (A32) gives, for every state Σ:

$$X^*(t)\, tr\left(\begin{bmatrix}\pi_1 & \pi_2\\ 0 & 0\end{bmatrix}\Sigma^{1/2}dB\right) = X^*(t)\, tr\left(\frac{1}{1-\gamma}\Theta'_{\nu^*}dB + \frac{\mathcal{D}\widehat{J}'}{\widehat{J}}\left(Y^{1/2}dBQ + Q'dB'Y^{1/2}\right)\right)$$

This implies (recall that $\Sigma^{1/2} = SY^{-1/2}$):

$$Y^{-1/2}S'\begin{bmatrix}\pi_1 & 0\\ \pi_2 & 0\end{bmatrix} = \frac{1}{1-\gamma}\Theta_{\nu^*} + 2Y^{1/2}AQ'$$

We conclude that the portfolio weight $\pi = (\pi_1, \pi_2)'$ is

$$\begin{aligned}
\pi &= \frac{1}{1-\gamma}\Sigma^{-1}\mu^e + 2\Sigma^{-1}SAQ'e_1 \\
&= \frac{1}{1-\gamma}\Sigma^{-1}\begin{pmatrix}\mu_1^e\\ \mu_2^e\end{pmatrix} + 2\Sigma^{-1}S\begin{pmatrix} q_{11}A_{11}+q_{12}A_{12}+q_{13}A_{13}\\ q_{11}A_{21}+q_{12}A_{22}+q_{13}A_{23}\\ q_{11}A_{31}+q_{12}A_{32}+q_{13}A_{33}\end{pmatrix}
\end{aligned}$$

This concludes the proof of Proposition 5.

Panel A	mean of returns	volatility of returns	volatility of volatility
US	0.1106	0.1597	0.0574
Germany	0.1350	0.2013	0.0856
Treasuries	0.0165	0.0906	0.0107
Aaa	0.0184	0.0857	0.0100

Panel B	mean of correlation	volatility of correlation
US-Germany	0.3672	0.1469
Treasuries-Aaa	0.8841	0.0953

Table 1: **Unconditional moments of asset returns.** *Panel A:* Unconditional mean, volatility and volatility of volatility of the S&P100 and the DAX index daily returns (first two rows) and of Treasury Bills and Aaa Lehmann corporate bond index daily returns (last two rows), for the periods January 1988 to December 2005 and April 1996 to December 2005, respectively. Unconditional volatilities of volatilities are estimated as the unconditional volatility of conditional volatilities estimated by Engle's (2002) Dynamic Conditional Correlations model. *Panel B:* Unconditional mean and volatility of correlations for the S&P100 and DAX index daily returns and for the Treasury and Aaa Lehmann corporate bond index daily returns, for the periods January 1988 to December 2005 and April 1996 to December 2005, respectively. Unconditional volatilities of correlations are estimated as the unconditional volatility of conditional correlations estimated by Engle's (2002) Dynamic Conditional Correlations model.

	M		Q	
US-Germany	-0.2816	0.0147	-0.0009	-0.0030
	0.0502	-0.1548	0.0028	0.0022
Treasury-Aaa	-0.4563	0.0273	-0.0007	0.0020
	0.0659	-0.3951	0.0024	0.0019

Table 2: **Calibrated parameters of the Wishart covariance matrix process.** Calibrated matrices M and Q for returns dynamics (1) under Wishart covariance matrix process (4) for $\Omega\Omega' = kQQ'$ and $k = 10$. Parameters have been calibrated to the estimated unconditional moments of returns and volatilities in Table 1 for US/German equity index returns and Treasury Bills/Aaa corporate bond index returns.

Inv. Horizon (Years)	*Correlation hedging US*	*Correlation hedging Germany*	*Volatility hedging US*	*Covariance hedging US volatility*	*Volatility hedging Germany*	*Covariance hedging Germany volatility*
0.5	-0.0128	-0.0033	-0.0005	-0.0015	-0.0013	-0.0016
1	-0.0170	-0.0074	-0.0012	-0.0034	-0.0028	-0.0030
2	-0.0287	-0.0187	-0.0034	-0.0086	-0.0061	-0.0065
3	-0.0461	-0.0356	-0.0070	-0.0163	-0.0100	-0.0118
4	-0.0717	-0.0605	-0.0133	-0.0277	-0.0144	-0.0201
5	-0.1086	-0.0963	-0.0242	-0.0442	-0.0188	-0.0320
6	-0.1404	-0.1301	-0.0329	-0.0672	-0.0227	-0.0487
7	-0.1904	-0.1786	-0.0550	-0.0884	-0.0251	-0.0613
8	-0.2456	-0.2213	-0.0899	-0.1183	-0.0301	-0.0882

Table 3: **The horizon effect.** We summarizes the optimal correlation and volatility hedging components, and the hedging demand against covariance risk due to volatility, for the international equity diversification scenario and for investment horizons from 6 months to 8 years. Parameters have been calibrated to the estimated unconditional moments of returns and volatilities of US/German equity index returns in Table 1.

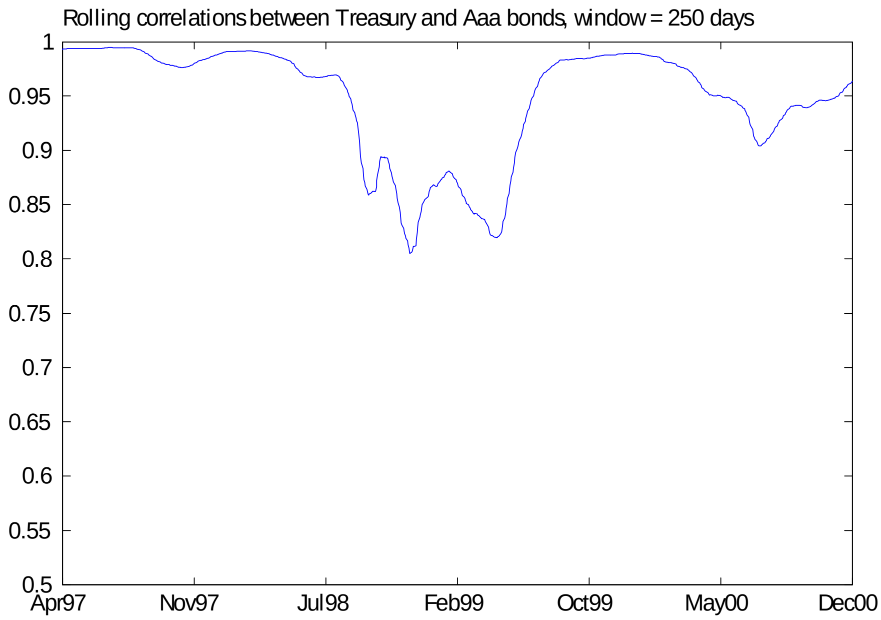


Figure 1. Rolling correlations of Treasury Bill and Aaa corporate bonds. The Figure presents correlations for Treasury Bill and Aaa daily bond index returns over the period April 1997 to December 2000. Dynamic correlations are estimated by a simple empirical correlation estimator based on a one year rolling window that is updated daily.

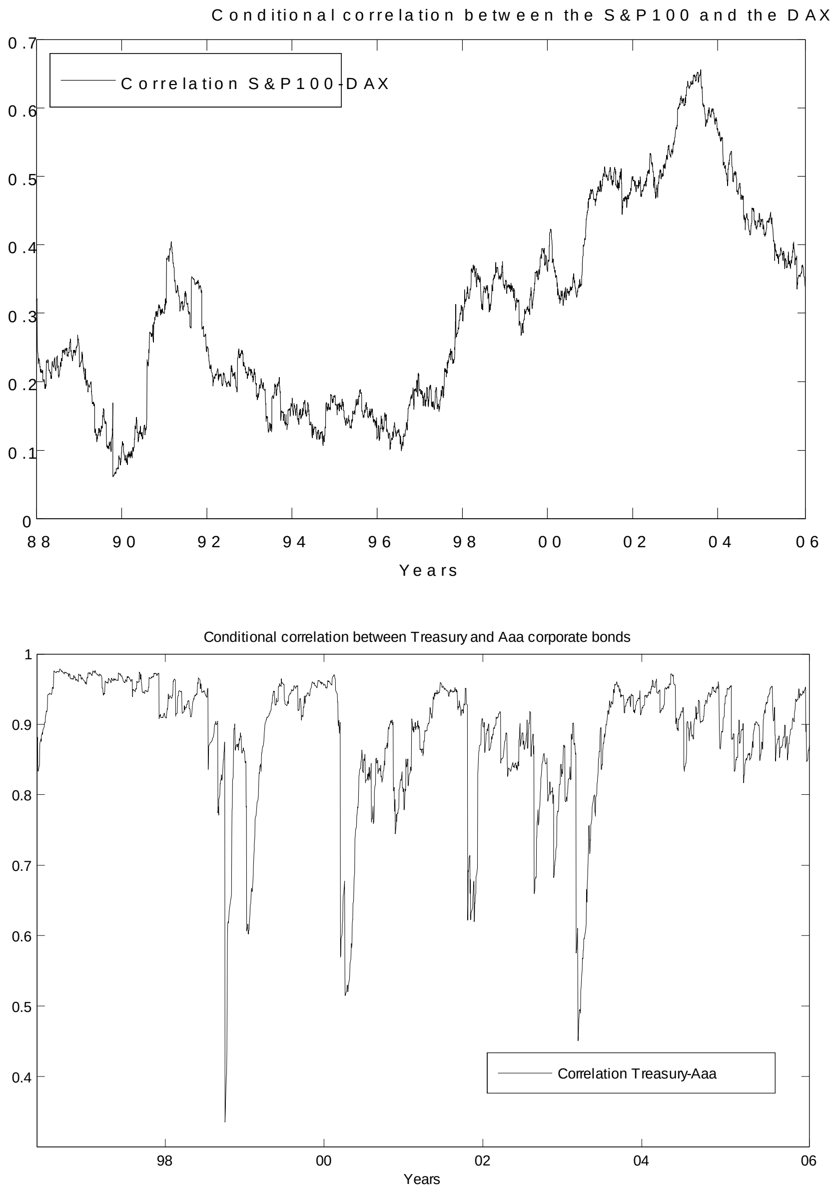


Figure 2. DCC conditional correlations. The upper panel presents estimated conditional correlations from Engle's (2002) DCC model for US and German daily equity index returns, over the period January 1988 to December 2005. The bottom panel presents estimated conditional correlations from Engle's (2002) DCC model for Treasury Bill and Aaa daily bond index returns, over the period April 1996 to December 2005.

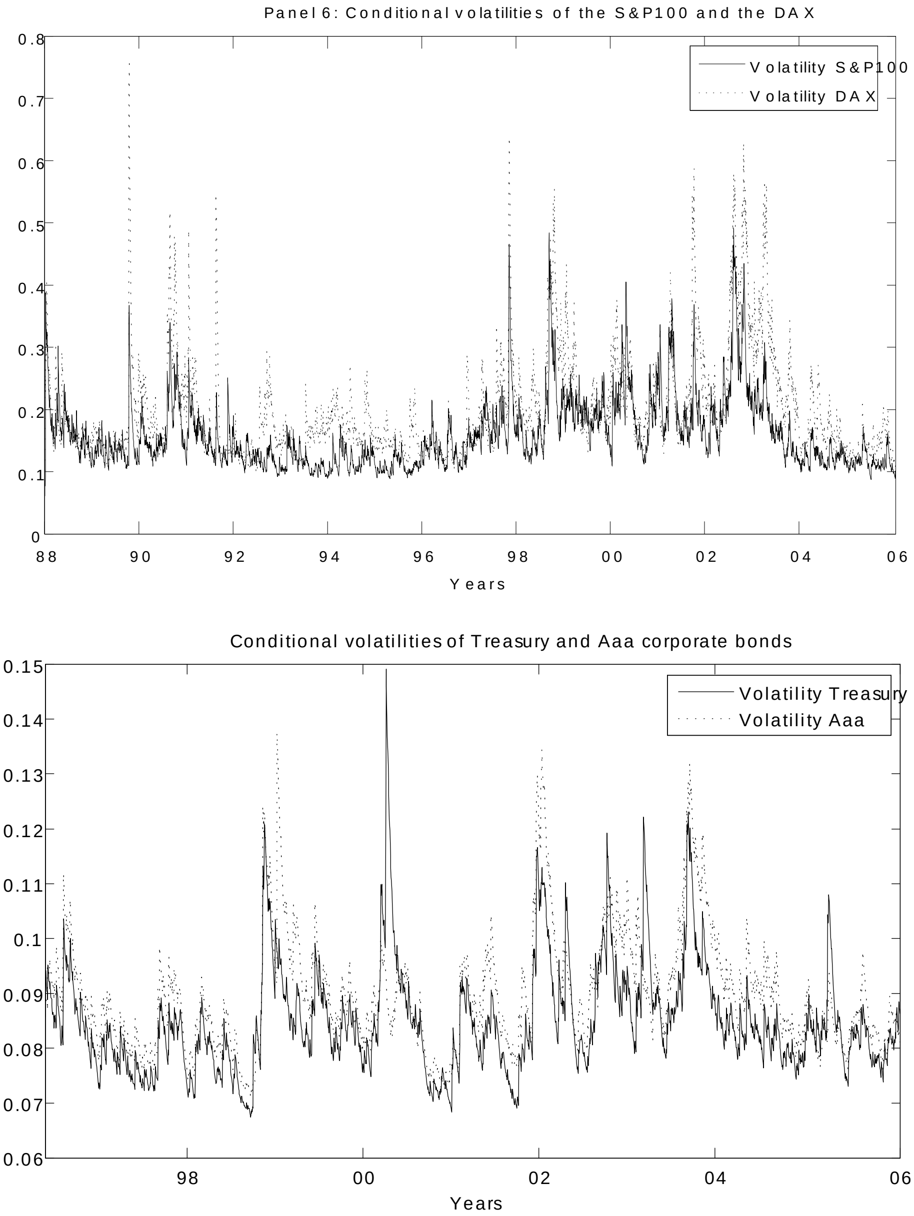


Figure 3. DCC conditional volatilities. The upper panel presents estimated conditional volatilities from Engle's (2002) DCC model for US and German daily equity index returns, over the period January 1988 to December 2005. The bottom panel presents estimated conditional volatilities from Engle's (2002) DCC model for Treasury Bill and Aaa daily bond index returns, over the period April 1996 to December 2005.

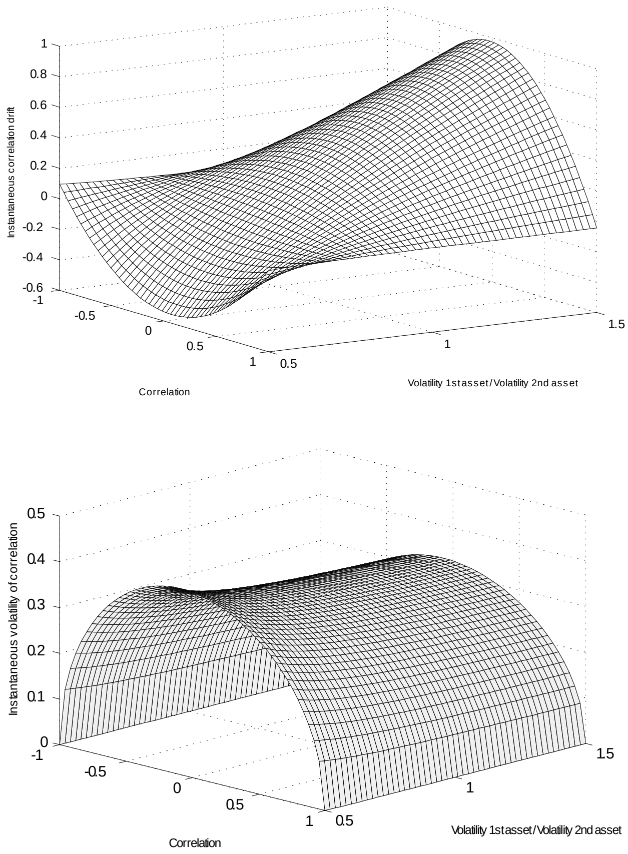
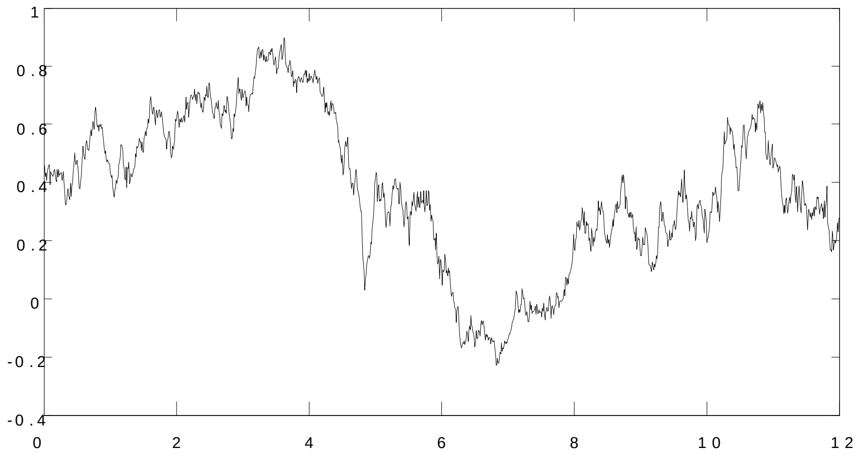


Figure 4. Drift and instantaneous volatility of correlation dynamics. The upper (bottom) panel presents the instantaneous drift (instantaneous volatility) of the correlation process implied by the diffusion process (4), as a function of the correlation level ρ and the volatility ratio $\sqrt{\Sigma_{11}/\Sigma_{22}}$. Model parameters have been calibrated to US and German daily equity index returns, over the period January 1988 to December 2005, as described in Section 4. For all plots, Σ_{22} has been fixed at the estimated level of German returns unconditional variance.



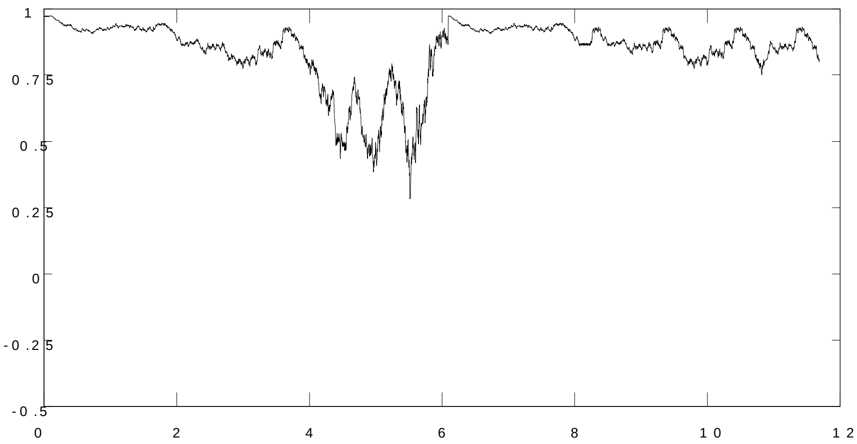


Figure 5. Correlation dynamics. The upper panel presents a simulated trajectory of conditional returns correlations from the Wishart dynamics (4), calibrated to the unconditional moments of US and German daily equity index returns over the period January 1988 to December 2005. The bottom panel presents a simulated trajectory of conditional returns correlations from the Wishart dynamics (4), calibrated to the unconditional moments of Treasury Bill and Aaa monthly bond index returns, over the period April 1996 to December 2005.

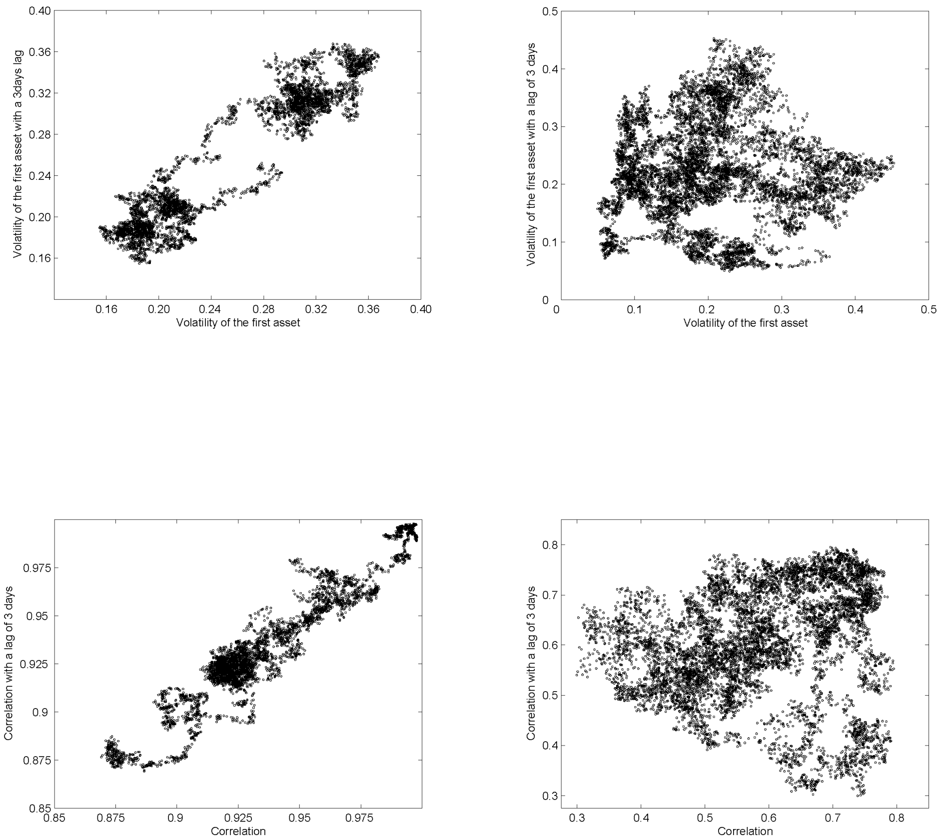


Figure 6. Volatility and correlation clustering. All panels present scatter plots of simulated daily volatilities and correlations of the first asset return, for two different parameterizations of the Wishart dynamics (4), plotted against their past values three days before. Panels on the left present a situation with high and positive persistence in volatility and correlation. Panels on the right present a situation with no persistence in volatilities and correlations.

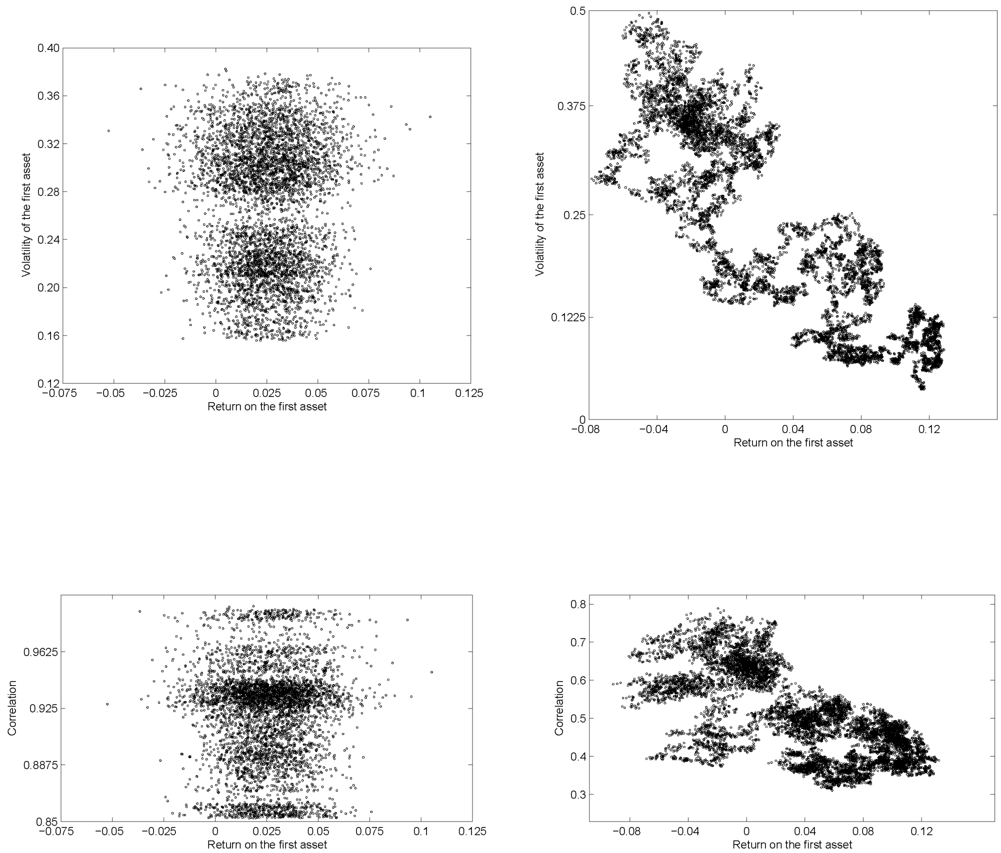


Figure 7. Volatility and correlation-leverage. The two top panels present scatter plots of returns volatilities of asset one, for the same parameterizations of Figure 6, plotted against the contemporaneous return of asset one. The two bottom panels present scatter plots of returns correlations, for the same parameterizations of Figure 6, plotted against the contemporaneous return of asset one. Panels on the left depict a situation with no volatility and correlation-leverage ffects with respect to the return of asset one. Panels on the right depict a situation with both volatility and correlation-leverage effects with respect to the return of asset one.

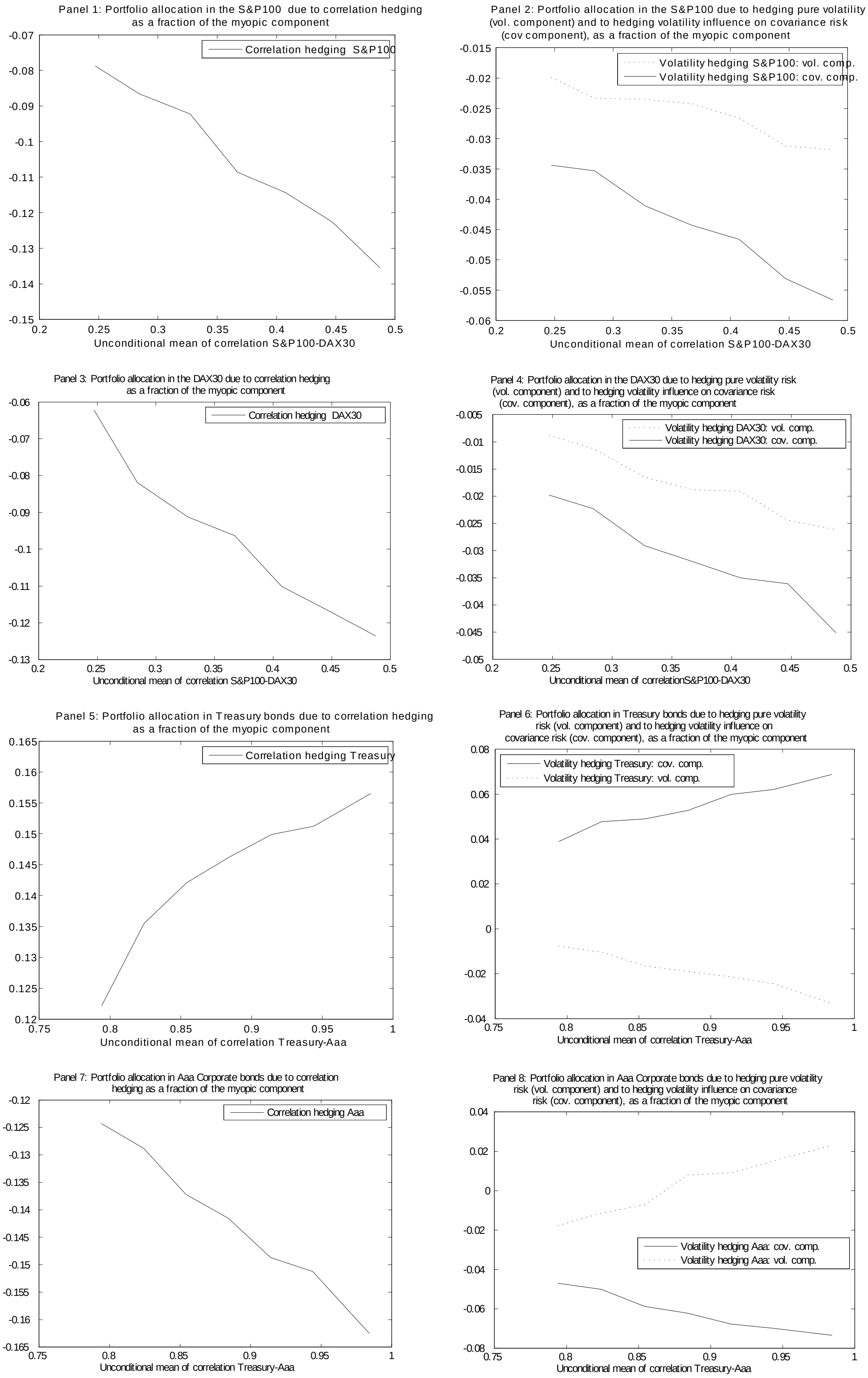


Figure 8. First calibration. Unconditional mean, standard deviation of returns volatilities and unconditional volatility of correlation are fixed at the DCC estimates. The correlation mean varies around the unconditional correlation estimate. Left panels plot correlation hedging as a fraction of the absolute myopic demands. Right panels plot volatility hedging as a fraction of the absolute myopic component. In Panels 1-4, the opportunity set is composed of the US and German equity indices, whereas in Panels 5-8 it is composed of US Treasury and Aaa Corporate bond indices. The investment horizon is 5 years. The relative risk aversion coefficient is $1 - \gamma = 3$.

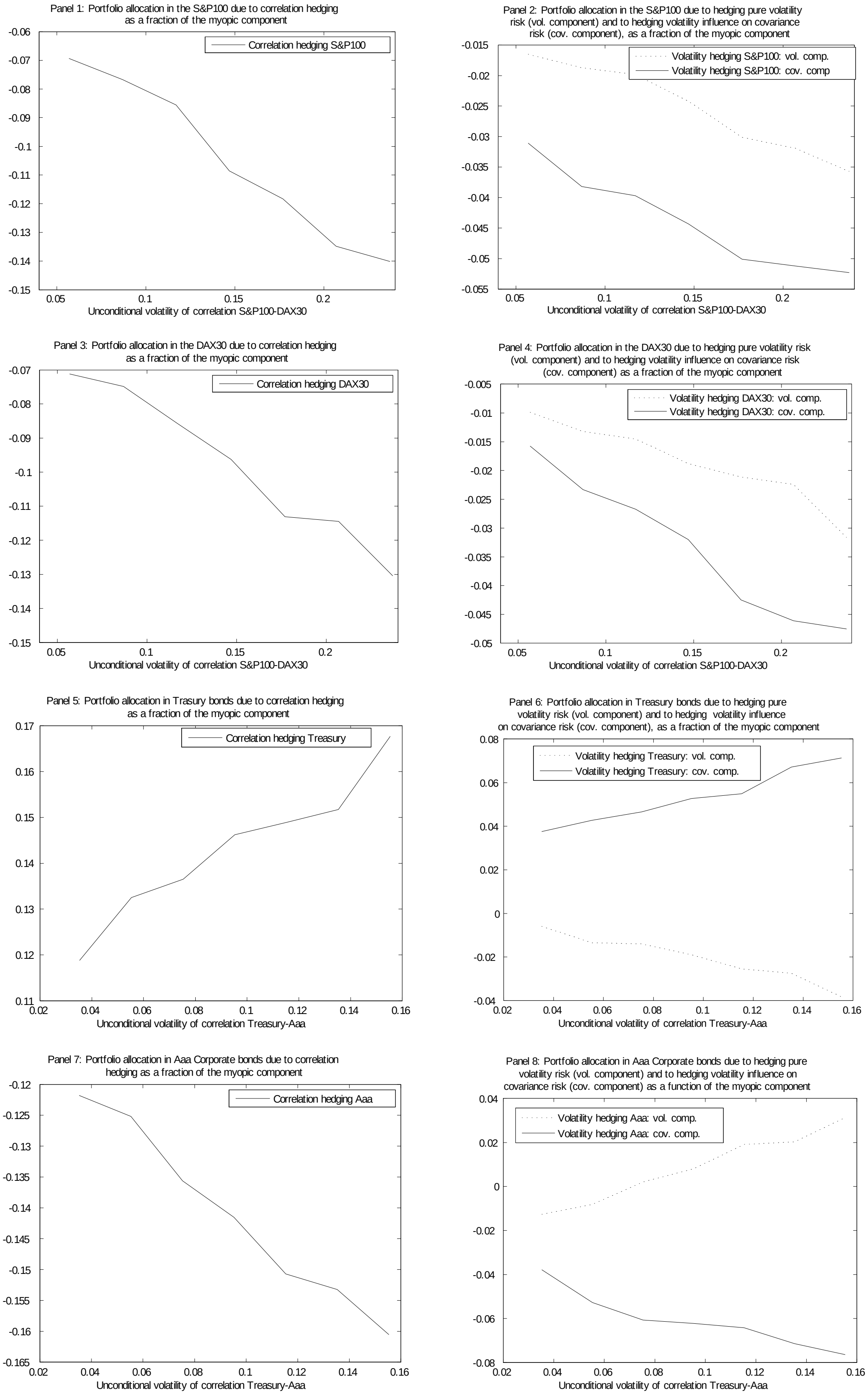


Figure 9. Second calibration. Unconditional mean, standard deviation of returns volatilities and unconditional mean of correlation are fixed at the DCC estimates. The correlation volatility varies around the unconditional correlation volatility implied by the DCC estimate. Left panels plot correlation hedging as a fraction of absolute myopic demands. Right panels plot volatility hedging as a fraction of the absolute myopic components. In Panels 1-4, the investment opportunity set is composed of US and German equity indices, whereas in Panels 5-8 it is composed of US Treasury and Aaa Corporate bond indices. The investment time-horizon is 5 years. The relative risk aversion coefficient is $1-\gamma = 3$.

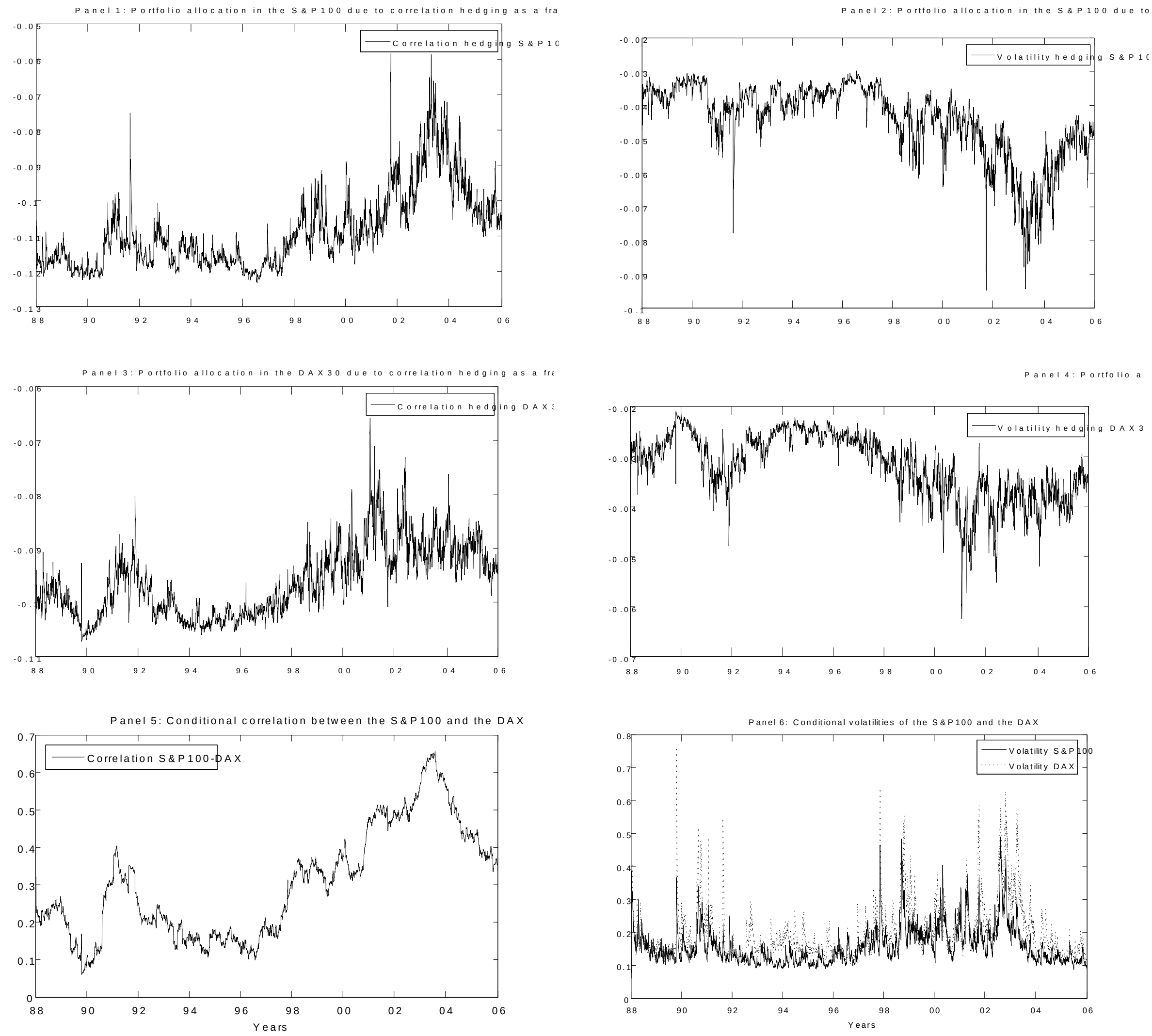


Figure 10. Conditional calibration for equities. Conditional portfolio policy for the 5-year time horizon due to correlation hedging (two top left panels) and volatility hedging (two top right panels), as a fraction of the corresponding absolute myopic allocation, for the US (Panels 1 and 3) and the German (Panels 2 and 4) equity indices. Panels 5 and 6 plot DCC estimates for the conditional correlation and the conditional volatilities of US and German stock index returns. The relative risk aversion coefficient used is $1 - \gamma = 3$.

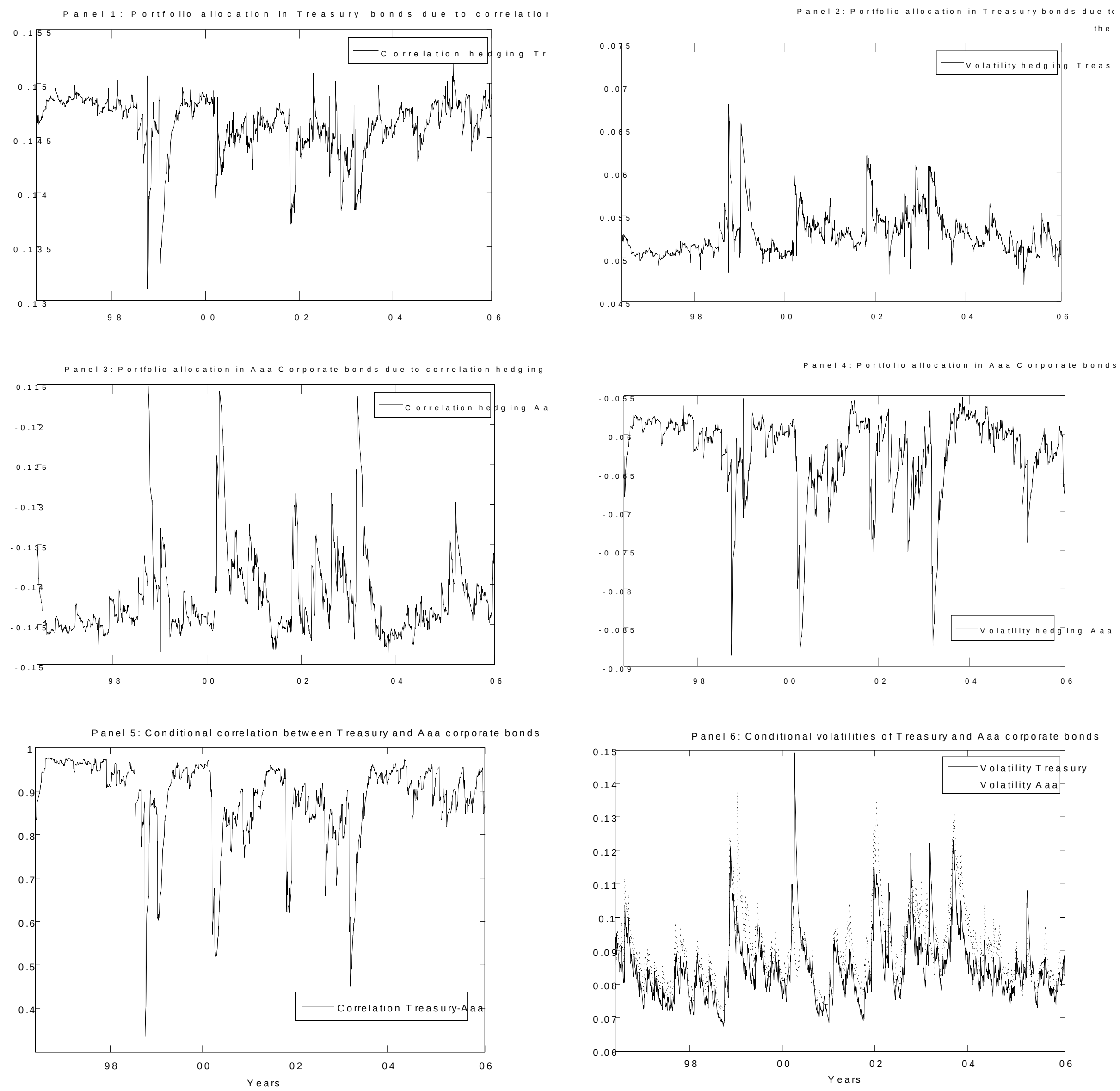


Figure 11. Conditional calibration for bonds. Conditional portfolio policy for the 5-year time horizon due to correlation hedging and volatility hedging, as a fraction of the corresponding absolute myopic allocation, for US Treasury bonds (Panels 1 and 2) and US Aaa corporate bonds (Panels 3 and 4). Panels 5 and 6 plot DCC estimates for the conditional correlation and the conditional volatilities of Treasury and Aaa corporate bond index returns. The relative risk aversion coefficient used is $1 - \gamma = 3$.

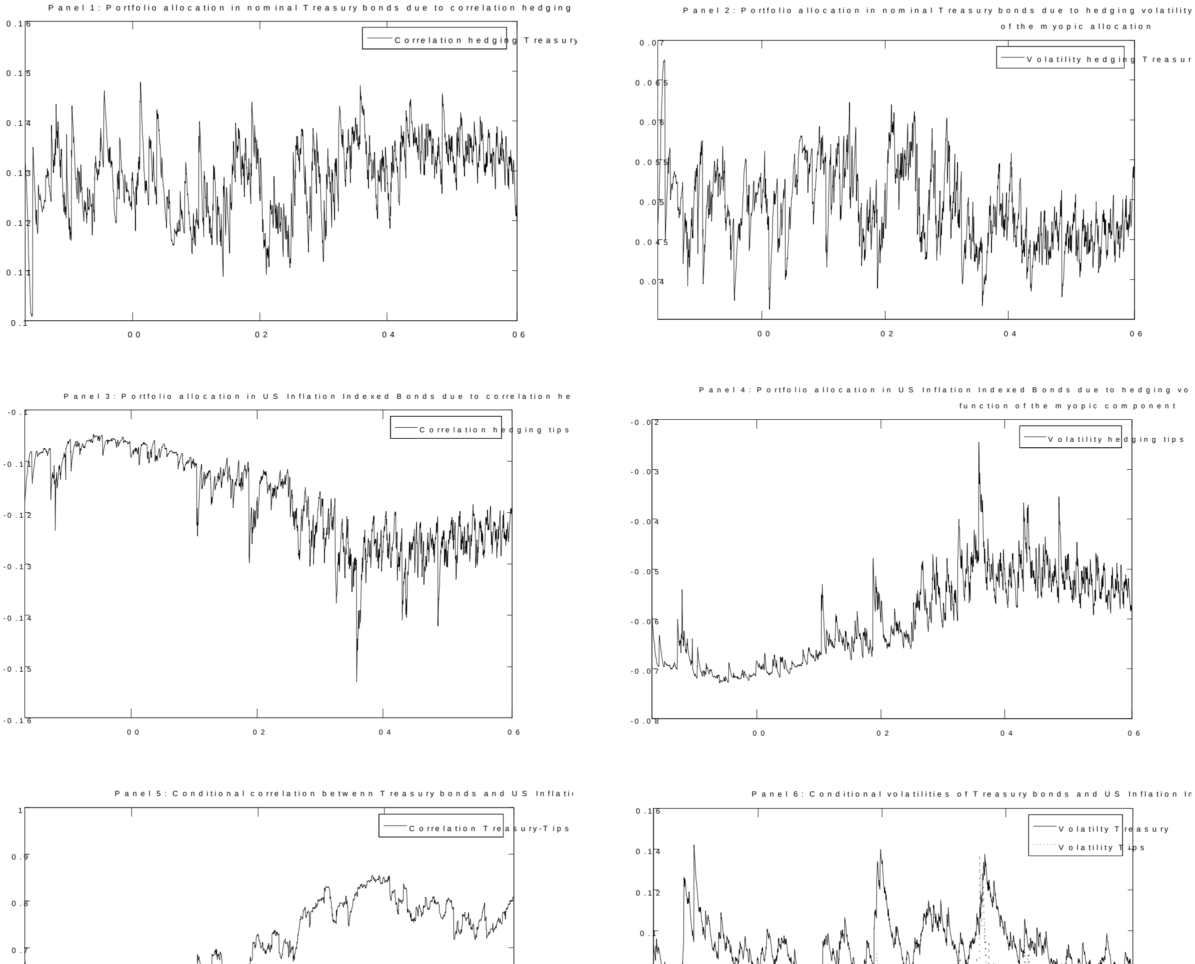


Figure 12. Conditional calibration for bonds. Conditional portfolio policy for the 5-year time horizon due to correlation hedging and volatility hedging, as a fraction of the corresponding absolute myopic allocation, for US Treasury bonds (Panels 1 and 2) and US Inflation Indexed bonds (Panels 3 and 4). Panels 5 and 6 plot DCC estimates for the conditional correlation and the conditional volatilities of Treasury and Inflation Indexed bond index returns. The relative risk aversion coefficient used is $1-\gamma=3$.

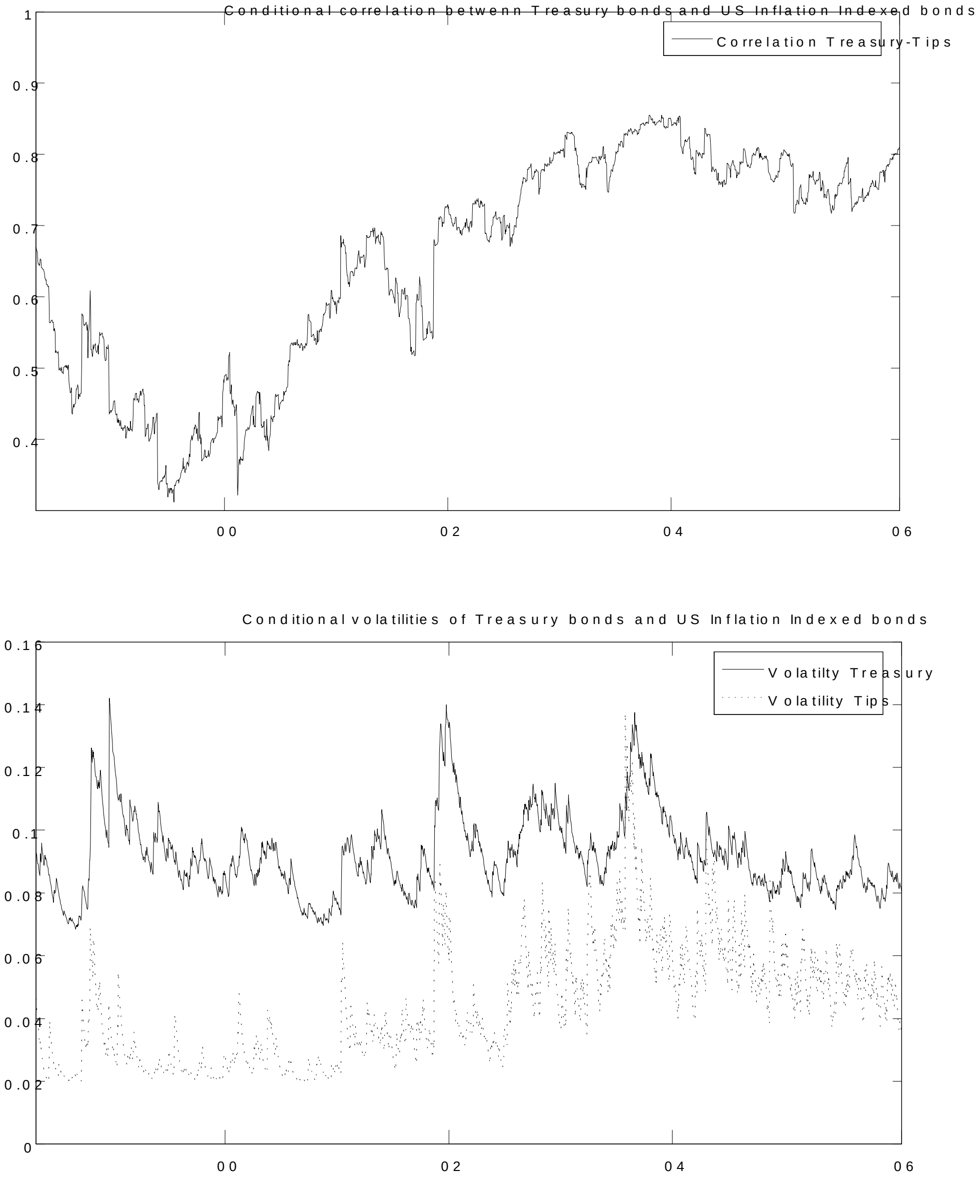


Figure 13. DCC conditional correlations. The upper panel presents estimated conditional correlations from Engle's (2002) DCC model for nominal Treasury and US Inflation Indexed bonds, over the period April 1998 to December 2005. The bottom panel presents estimated conditional volatilities from Engle's (2002) DCC model for the same dataset.